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Analytic solutions for marginal deformations in open string field theory

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Abstract: We develop a calculable analytic approach to marginal deformations in open string field theory using wedge states with operator insertions. For marginal operators with regular operator products, we construct analytic solutions to all orders in the deformation parameter. In particular, we construct an exact time-dependent solution that describes D-brane decay and incorporates all α' corrections. For marginal operators with singular operator products, we construct solutions by regularizing the singularity and adding counterterms. We explicitly carry out the procedure to third order in the deformation parameter.

Keywords: String Field Theory, Bosonic Strings.

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Contents

1. Introduction

Mapping the landscape of vacua is one of the outstanding challenges in string theory. A simpler version of the problem is to characterize the "open string landscape," the set of possible D-brane configurations in a fixed closed string background. In recent years evidence has accumulated that classical open string field theory (OSFT) gives an accurate description of the open string landscape. See [1 − 3] for reviews. Much of this evidence is based on numerical work in level truncation, and there remain many interesting questions. Is the correspondence between boundary conformal field theories and classical OSFT solutions one-to-one? Is the OSFT action of a single D-brane capable of describing configurations of multiple D-branes? Answering these questions is likely to require analytic tools. Important analytic progress was made by Schnabl [4]. He found the exact solution corresponding to the tachyon vacuum by exploiting the simplifications coming from the clever gauge fixing condition

$$B\Psi = 0\,, \tag{1.1}$$

where B is the antighost zero mode in the conformal frame of the sliver. Various aspects of Schnabl's construction have been studied in [5 − 12].

In this paper we describe new analytic solutions of OSFT corresponding to exactly marginal deformations of the boundary conformal field theory (CFT). Previous work on exactly marginal deformations in OSFT [13] was based on solving the level-truncated equations of motion in Siegel gauge. The level-truncated string field was determined as a function of the vacuum expectation value of the exactly marginal mode fixed to an arbitrary finite value. Level truncation lifts the flat direction, but it was seen that as the level is increased the flat direction is recovered with better and better accuracy. Instead, our approach is to expand the solution as $\Psi_\lambda = \sum_{n=1}^{\infty} \lambda^n \Psi^{(n)}$, where λ parameterizes the exact flat direction. We solve the equation of motion recursively to find an analytic expression for $\Psi^{(n)}$. Our results are exact in that we are solving the full OSFT equation of motion, but they are perturbative in λ; by contrast, the results of [13] are approximate since the equation of motion has been level-truncated, but they are non-perturbative in the deformation parameter.

The perturbative approach of this paper has certainly been attempted earlier using the Siegel gauge. Analytic work, however, is out of the question because in the Siegel gauge the Riemann surfaces associated with $\Psi^{(n)}$, with $n > 2$, are very complicated. The new insight that makes the problem tractable is to use, as in [4], the remarkable properties of wedge states with insertions [14 − 16].

We find qualitatively different results, according to whether the matter vertex operator V that generates the deformation has regular or singular operator products. Sections 2 and 3 of the paper are devoted to the case of regular operator products, and the case of singular operator products is discussed in section 4. A key technical point is the calculation of the action of B/L, where $L = \{ Q_B, B \}$, on products of string fields.

If V has regular operator products, the equation of motion can be systematically solved in the Schnabl gauge (1.1). The solution takes a strikingly compact form given in the CFT language by (3.3), and its geometric picture is presented in figure 1. The solution $\Psi^{(n)}$ is made of a wedge state with n insertions of cV on its boundary. The relative separations of the boundary insertions are specified by $n-1$ moduli $\{t_i\}$, with $0 \leq t_i \leq 1$, which are to be integrated over. Each modulus is accompanied by an antighost line integral $\mathcal{B}$. The explicit evaluation of $\Psi^{(n)}$ in the level expansion is straightforward for a specific choice of V.

In section 3.2 we apply this general result to the operator $V = e^{\frac{1}{\sqrt{\alpha'}} X^0}$ [17 − 23]. This deformation describes a time-dependent tachyon solution that starts at the perturbative vacuum in the infinite past and (if $\lambda < 0$) begins to roll toward the non-perturbative vacuum. The parameter λ can be rescaled by a shift of the origin of time, so the solutions are physically equivalent. The time-dependent tachyon field takes the form

$$T(x^0) = \lambda \, e^{\frac{1}{\sqrt{\alpha'}} x^0} + \sum_{n=2}^{\infty} \lambda^n \, \beta_n \, e^{\frac{1}{\sqrt{\alpha'}} n x^0} \, . \tag{1.2}$$

We derive a closed-form integral expression for the coefficients β_n and evaluate them numerically. We find that the coefficients decay so rapidly as n increases that it is plausible that the solution is absolutely convergent for any value of x^0. Our exact result confirms the surprising oscillatory behavior found in the p-adic model [19] and in level-truncation studies of OSFT [19, 23]. The tachyon (1.2) overshoots the non-perturbative vacuum and oscillates with ever-growing amplitude. It has been argued that a field redefinition to the variables of boundary SFT would map this oscillating tachyon to a tachyon field monotonically relaxing to the non-perturbative vacuum [23]. It would be very interesting to calculate the pressure of our exact solution and check whether it tends to zero in the infinite future, as would be expected from Sen's analysis of tachyon matter [24, 1].

In section 3.3 we consider the lightcone vertex operator ∂X^+, another example of a marginal vertex operator with regular operator products. Following [25], we construct the string field solution inspired by the Born-Infeld solution that describes a fundamental string ending on a D-brane [26]. The lightcone direction X^+ is a linear combination of the time direction and a direction normal to the brane, and the vertex operator is dressed by $A(k_i)\, e^{ik_i X^i}$ and integrated over the momenta k_i along the spatial directions on the brane. The solution is not fully self-contained within open string field theory: it requires sources, which makes the analysis delicate. Sources are also required in the Born-Infeld description of the solution.

If the operator product expansion (OPE) of V with itself is $V(z)V(0) \sim 1/z^2$, the solution presented in figure 1 is not well defined because divergences arise as the separations t_i of the boundary insertions go to zero. We study the required modifications in section 4. An important example is the Wilson-line deformation ∂X. We regularize the divergences by imposing a cut-off in the integration region of the moduli. It turns out that counterterms can be added to obtain $\Psi^{(2)}$ that is finite and satisfies the equation of motion. Surprisingly, the result necessarily violates the gauge condition (1.1)! The naive solution $\Psi^{(2)} = -\frac{B}{L}(\Psi^{(1)} * \Psi^{(1)})$ breaks down because the string field $\Psi^{(1)} * \Psi^{(1)}$ contains a component in the kernel of L. This phenomenon is a peculiar quirk of Schnabl gauge that has no counterpart in Siegel gauge. Due to this technical complication, the construction of the higher $\Psi^{(n)}$ becomes quite cumbersome, though still simpler than in Siegel gauge. We argue that for all n, appropriate counterterms can be added to achieve a finite $\Psi^{(n)}$ that solves the equation of motion. We discuss in detail the case of $\Psi^{(3)}$ and verify the nontrivial cancellations that must occur for the construction to succeed. We leave it for future work to achieve simpler closed-form expressions for $\Psi^{(n)}$. Such expressions will be needed to investigate the nature of the perturbative series in λ and to make contact with the non-perturbative, but approximate, level-truncation results of [13]. It will also be interesting to understand better the relation between the conditions for exact marginality of boundary CFT [27] and the absence of obstructions in solving the equation of motion of string field theory. The technology developed in this paper will be also useful in open superstring field theory [28].

Independent work by M. Schnabl on the subject of marginal deformations in string field theory appears in [29].

2. The action of B/L

2.1 Solving the equation of motion in the Schnabl gauge

For any matter primary field V of dimension one, the state $\Psi^{(1)}$ corresponding to the operator $cV(0)$ is BRST closed:

$$Q_B \Psi^{(1)} = 0\,. \tag{2.1}$$

In the context of string field theory, this implies that the linearized equation of motion of string field theory is satisfied. When the marginal deformation associated with V is *exactly* marginal, we expect that a solution of the form

$$\Psi_\lambda = \sum_{n=1}^{\infty} \lambda^n \, \Psi^{(n)} \,, \tag{2.2}$$

where λ is a parameter, solves the nonlinear equation of motion

$$Q_B \Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0 \,. \tag{2.3}$$

The equation that determines $\Psi^{(n)}$ for $n > 1$ is

$$Q_B \Psi^{(n)} = \Phi^{(n)} \quad \text{with} \quad \Phi^{(n)} = -\sum_{k=1}^{n-1} \Psi^{(n-k)} * \Psi^{(k)} \,. \tag{2.4}$$

For this equation to be consistent, $\Phi^{(n)}$ must be BRST closed. This is easily shown using the equations of motion at lower orders. For example,

$$Q_B \Phi^{(2)} = -Q_B \left(\Psi^{(1)} * \Psi^{(1)} \right) = -Q_B \Psi^{(1)} * \Psi^{(1)} + \Psi^{(1)} * Q_B \Psi^{(1)} = 0 \tag{2.5}$$

when $Q_B \Psi^{(1)} = 0$. It is crucial that $\Phi^{(n)}$ be BRST *exact* for all $n > 1$, or else we would encounter an obstruction in solving the equations of motion. No such obstruction is expected to arise if the matter operator V is exactly marginal, so we can determine $\Psi^{(n)}$ recursively by solving $Q_B \Psi^{(n)} = \Phi^{(n)}$. This procedure is ambiguous as we can add any BRST-closed term to $\Psi^{(n)}$, so we need to choose some prescription. A traditional choice would be to work in Siegel gauge. The solution $\Psi^{(n)}$ is then given by acting with b_0/L_0 on $\Phi^{(n)}$. In practice this is cumbersome since the combination of star products and operators b_0/L_0 in the Schwinger representation generates complicated Riemann surfaces in the CFT formulation.

Inspired by Schnabl's success in finding an analytic solution for tachyon condensation, it is natural to look for a solution Ψ_λ in the Schnabl gauge:

$$B \Psi_\lambda = 0 \,. \tag{2.6}$$

Our notation is the same as in [5, 7, 8]. In particular the operators B and L are the zero modes of the antighost and of the energy-momentum tensor T, respectively, in the conformal frame of the sliver,[1]

$$B \equiv \oint \frac{d\xi}{2\pi i} \frac{f(\xi)}{f'(\xi)} \, b(\xi) \,, \quad L \equiv \oint \frac{d\xi}{2\pi i} \frac{f(\xi)}{f'(\xi)} \, T(\xi) \,, \quad f(\xi) \equiv \frac{2}{\pi} \arctan(\xi) \,. \tag{2.7}$$

We define $L^\pm \equiv L \pm L^\star$ and $B^\pm \equiv B \pm B^\star$, where the superscript $\star$ indicates BPZ conjugation, and we denote with subscripts L and R the left and right parts, respectively, of these operators. Formally, a solution of (2.4) obeying (2.6) can be constructed as follows:

$$\Psi^{(n)} = \frac{B}{L} \Phi^{(n)} \,. \tag{2.8}$$

[1]Using reparameterizations, as in [8], it should be straightforward to generalize the discussion to general projectors. In this paper we restrict ourselves to the simplest case of the sliver.

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This can also be written as

$$\Psi^{(n)} = \int_0^\infty dT\, B e^{-TL}\, \Phi^{(n)}\,, \tag{2.9}$$

if the action of e^{-TL} on $\Phi^{(n)}$ vanishes in the limit $T \to \infty$. It turns out that the action of B/L on $\Phi^{(n)}$ is not always well defined. As we discuss in detail in section 4, if the matter primary field V has a singular OPE with itself, the formal solution breaks down and the required modification necessarily violates the gauge condition (2.6). On the other hand, if operator products of the matter primary field are regular, the formal solution is well defined, as we will confirm later. In the rest of this section, we study the expression (2.9) for $n = 2$ in detail.

2.2 Algebraic preliminaries

We prepare for our work by reviewing and deriving some useful algebraic identities. For further details and conventions the reader can refer to [7, 8].

An important role will be played by the operator $L - L_L^+$ and the antighost analog $B - B_L^+$. These operators are derivations of the star algebra. This is seen by writing the first one, for example, as a sum of two familiar derivations in the following way:

$$L - L_L^+ = \frac{1}{2}L^- + \frac{1}{2}(L_R^+ + L_L^+) - L_L^+ = \frac{1}{2}L^- + \frac{1}{2}(L_R^+ - L_L^+) = \frac{1}{2}(L^- + K)\,. \tag{2.10}$$

We therefore have

$$(L - L_L^+)\,(\phi_1 * \phi_2) = (L - L_L^+)\,\phi_1 * \phi_2 + \phi_1 * (L - L_L^+)\phi_2\,. \tag{2.11}$$

Noting that $L_L^+\,(\phi_1 * \phi_2) = L_L^+\,\phi_1 * \phi_2$, we find

$$L(\phi_1 * \phi_2) = L\phi_1 * \phi_2 + \ \phi_1 * (L - L_L^+)\,\phi_2\,, \tag{2.12}$$

$$B(\phi_1 * \phi_2) = B\phi_1 * \phi_2 + (-1)^{\phi_1}\phi_1 * (B - B_L^+)\phi_2\,. \tag{2.13}$$

Here and in what follows, a string field in the exponent of -1 denotes its Grassmann property: it is 0 mod 2 for a Grassmann-even string field and 1 mod 2 for a Grassmann-odd string field. From (2.12) and (2.13) we immediately deduce formulas for products of multiple string fields. For B, for example, we have

$$B(\phi_1 * \phi_2 * \ldots \phi_n) = (B\phi_1) * \ldots * \phi_n + \sum_{m=2}^{n} (-1)^{\sum_{k=1}^{m-1}\phi_k}\, \phi_1 * \ldots * (B - B_L^+)\phi_m * \ldots * \phi_n\,. \tag{2.14}$$

Exponentiation of (2.12) gives

$$e^{-TL}(\phi_1 * \phi_2) = e^{-TL}\phi_1 * e^{-T(L - L_L^+)}\phi_2\,. \tag{2.15}$$

From the familiar commutators

$$[L, L^+] = L^+\,, \quad [B, L^+] = B^+\,, \tag{2.16}$$

we deduce

$$[L, L_L^+] = L_L^+\,, \quad [B, L_L^+] = B_L^+\,. \tag{2.17}$$

See section 2 of [7] for a careful analysis of this type of manipulations. We will need to reorder exponentials of the derivation $L - L_L^+$. We claim that

$$e^{-T(L-L_L^+)} = e^{(1-e^{-T})L_L^+}\, e^{-TL}\,. \tag{2.18}$$

The above is a particular case of the Baker-Campbell-Hausdorff formula for a two-dimensional Lie algebra with generators x and y and commutation relation $[\,x, y\,] = y$. In the adjoint representation we can write

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{2.19}$$

It follows that as two-by-two matrices, $x^2 = x$, $xy = y$, $yx = 0$, and $y^2 = 0$. One then verifies that

$$e^{\alpha x + \beta y} = e^{\frac{\beta}{\alpha}(e^{\alpha}-1)y}\, e^{\alpha x} \quad \text{when} \quad [\,x, y\,] = y\,. \tag{2.20}$$

With $\alpha = -\beta = -T$, $x = L$, and $y = L_L^+$, (2.20) reproduces (2.18).

2.3 The action of B/L and its geometric interpretation

We are now ready to solve the equation for $\Psi^{(2)}$. The state $\Psi^{(1)}$ satisfies

$$Q_B\Psi^{(1)} = 0\,, \qquad B\Psi^{(1)} = 0\,, \qquad L\Psi^{(1)} = 0\,. \tag{2.21}$$

We will use correlators in the sliver frame to represent states made of wedge states and operator insertions. The state $\Psi^{(1)}$ can be described as follows:

$$\langle\, \phi, \Psi^{(1)}\,\rangle = \langle\, f \circ \phi(0)\, cV(1)\,\rangle_{\mathcal{W}_1}\,. \tag{2.22}$$

Note that cV is a primary field of dimension zero so that there is no associated conformal factor. Here and in what follows we use ϕ to denote a generic state in the Fock space and $\phi(0)$ to denote its corresponding operator. The surface $\mathcal{W}_\alpha$ is the one associated with the wedge state W_α in the sliver conformal frame. We use the doubling trick in calculating correlators. We define the oriented straight lines $V_\alpha^\pm$ by

$$\begin{gathered} V_\alpha^\pm = \Big\{\, z \,\Big|\, \mathrm{Re}(z) = \,\pm\frac{1}{2}\,(1+\alpha)\,\Big\}\,, \\ \text{orientation}\,:\, \pm\frac{1}{2}\,(1+\alpha) - i\,\infty \to\, \pm\frac{1}{2}\,(1+\alpha) + i\,\infty\,. \end{gathered} \tag{2.23}$$

The surface $\mathcal{W}_\alpha$ can be represented as the region between V_0^- and $V_{2\alpha}^+$, where V_0^- and $V_{2\alpha}^+$ are identified by translation.

A formal solution to the equation $Q_B\Psi^{(2)} = -\Psi^{(1)} * \Psi^{(1)}$ is

$$\Psi^{(2)} = -\int_0^\infty dT\, Be^{-TL}\,\Big[\,\Psi^{(1)} * \Psi^{(1)}\Big]\,. \tag{2.24}$$

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By construction, $B\Psi^{(2)} = 0$. Using the identities (2.15) and (2.13), we have

$$\Psi^{(2)} = -\int_0^\infty dT \left[B\, e^{-TL}\, \Psi^{(1)} * e^{-T(L-L_L^+)}\, \Psi^{(1)} - e^{-TL}\, \Psi^{(1)} * (B - B_L^+)\, e^{-T(L-L_L^+)}\, \Psi^{(1)} \right] . \tag{2.25}$$

Because of the properties of $\Psi^{(1)}$ in (2.21), the first term vanishes and the second reduces to

$$\Psi^{(2)} = \int_0^\infty dT\, \Psi^{(1)} * (B - B_L^+)\, e^{-T(L-L_L^+)}\, \Psi^{(1)} . \tag{2.26}$$

We further use the identity (2.18) together with $L\Psi^{(1)} = 0$ to find

$$\Psi^{(2)} = \int_0^\infty dT\, \Psi^{(1)} * (B - B_L^+)\, e^{(1-e^{-T})L_L^+}\, \Psi^{(1)} . \tag{2.27}$$

It follows from $[\, B, L_L^+\,] = B_L^+$ that $[\, B,\, g(L_L^+)\,] = B_L^+\, g'(L_L^+)$ for any analytic function g. Using this formula with $B\Psi^{(1)} = 0$, we find

$$\Psi^{(2)} = -\int_0^\infty dT\, e^{-T}\, \Psi^{(1)} * e^{(1-e^{-T})L_L^+}\, B_L^+ \Psi^{(1)} . \tag{2.28}$$

Using the change of variables $t = e^{-T}$, we obtain the following final expression of $\Psi^{(2)}$:

$$\Psi^{(2)} = \int_0^1 dt\, \Psi^{(1)} * e^{-(t-1)L_L^+}\, (-B_L^+)\, \Psi^{(1)} . \tag{2.29}$$

There is a simple geometric picture for $\Psi^{(2)}$. Let us represent $\langle\, \phi, \Psi^{(2)}\,\rangle$ in the CFT formulation. The exponential action of L_L^+ on a generic string field A can be written as

$$e^{-\alpha L_L^+} A = e^{-\alpha L_L^+} (\mathcal{I} * A) = e^{-\alpha L_L^+} \mathcal{I} * A = W_\alpha * A . \tag{2.30}$$

Here we have recalled the familiar expression of the wedge state $W_\alpha = e^{-\frac{\alpha}{2} L^+} \mathcal{I} = e^{-\alpha L_L^+} \mathcal{I}$ [4], where $\mathcal{I}$ is the identity string field. We thus learn that $e^{-\alpha L_L^+}$ with $\alpha > 0$ creates a semi-infinite strip with a width of α in the sliver frame, while $e^{-\alpha L_L^+}$ with $\alpha < 0$ deletes a semi-infinite strip with a width of $|\alpha|$. The inner product $\langle\, \phi, \Psi^{(2)}\,\rangle$ is thus represented by a correlator on $\mathcal{W}_{2-|t-1|} = \mathcal{W}_{1+t}$. In other words, the integrand in (2.29) is made of the wedge state W_{1+t} with operator insertions. The state ϕ is represented by the region between V_0^- and V_0^+ with the operator insertion $f \circ \phi(0)$ at the origin. The left factor of $\Psi^{(1)}$ in (2.29) can be represented by the region between V_0^+ and V_2^+ with an insertion of cV at $z = 1$. For $t = 1$ the right factor of $\Psi^{(1)}$ can be represented by the region between V_2^+ and V_4^+ with an insertion of cV at $z = 2$. For $0 < t < 1$, the region is shifted to the one between $V_{2-2|t-1|}^+ = V_{2t}^+$ and $V_{4-2|t-1|}^+ = V_{2+2t}^+$, and the insertion of cV is at $z = 2 - |t-1| = 1 + t$. Finally, the operator $(-B_L^+)$ is represented by an insertion of $\mathcal{B}$ [8] defined by

$$\mathcal{B} = \int \frac{dz}{2\pi i}\, b(z) , \tag{2.31}$$

where the contour of the integral can be taken to be $-V_\alpha^+$ with $1 < \alpha < 1+2t$. We thus have

$$\langle\, \phi, \Psi^{(2)}\,\rangle = \int_0^1 dt\, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t)\, \rangle_{\mathcal{W}_{1+t}} . \tag{2.32}$$

As $t \to 0$ the pair of cV's collide, and at $t = 1$ they attain the maximum separation.

The state $\Psi^{(2)}$ should formally solve the equation of motion by construction. Let us examine the BRST transformation of $\Psi^{(2)}$ more carefully based on the expression (2.32). The BRST operator in $\langle \phi, Q_B \Psi \rangle$ can be represented as an integral of the BRST current on $V^+_{2(1+t)} - V^+_0$:[2]

$$\langle \phi, Q_B \Psi^{(2)} \rangle = \int_0^1 dt \left\langle f \circ \phi(0) \int_{-V_0^+ + V^+_{2(1+t)}} \frac{dz}{2\pi i}\, j_B(z)\; cV(1)\, \mathcal{B}\, cV(1+t) \right\rangle_{\mathcal{W}_{1+t}} , \tag{2.33}$$

where j_B is the BRST current. Since cV is BRST closed, the only nontrivial action of the BRST operator is to change the insertion of the antighost to that of the energy-momentum tensor:

$$\langle \phi, Q_B \Psi^{(2)} \rangle = - \int_0^1 dt \, \langle f \circ \phi(0)\, cV(1)\, \mathcal{L}\, cV(1+t) \rangle_{\mathcal{W}_{1+t}} , \tag{2.34}$$

where

$$\mathcal{L} = \int \frac{dz}{2\pi i}\, T(z) , \tag{2.35}$$

and the contour of the integral can be taken to be $-V^+_\alpha$ with $1 < \alpha < 1 + 2t$. The minus sign on the right-hand side of (2.34) is from anticommuting the BRST current with the left cV. Since $\partial_t\, e^{-tL^+_L} = -L^+_L\, e^{-tL^+_L}$ and $-L^+_L$ corresponds to $\mathcal{L}$ in the correlator, an insertion of $\mathcal{L}$ is equivalent to taking a derivative with respect to t [5]. We thus find

$$\langle \phi, Q_B \Psi^{(2)} \rangle = - \int_0^1 dt \, \frac{\partial}{\partial t} \langle f \circ \phi(0)\, cV(1)\, cV(1+t) \rangle_{\mathcal{W}_{1+t}} . \tag{2.36}$$

The surface term from $t = 1$ gives $-\Psi^{(1)} * \Psi^{(1)}$. The equation of motion is therefore satisfied if the surface term from $t = 0$ vanishes. The surface term from $t = 0$ vanishes if

$$\lim_{t \to 0} cV(0)\, cV(t) = 0 . \tag{2.37}$$

Therefore, $\Psi^{(2)}$ defined by (2.32) does solve the equation $Q_B \Psi^{(2)} + \Psi^{(1)} * \Psi^{(1)} = 0$ when V satisfies (2.37). Since $\Psi^{(1)} * \Psi^{(1)}$ is a finite state, the equation guarantees that $Q_B \Psi^{(2)}$ is also finite. However, it is still possible that $\Psi^{(2)}$ has a divergent term which is BRST closed. The ghost part of $\Psi^{(2)}$ is finite since it is given by an integral of ψ_t over t from $t = 0$ to $t = 1$, where ψ_n is the key ingredient in the tachyon vacuum solution [4]:

$$\langle \phi, \psi_n \rangle = \langle f \circ \phi(0)\, c(1)\, \mathcal{B}\, c(1+n) \rangle_{\mathcal{W}_{1+n}} , \tag{2.38}$$

and the contour of the integral for $\mathcal{B}$ can be taken to be $-V_\alpha$ with $1 < \alpha < 2n + 1$. When the operator product of V with itself is regular, the condition (2.37) is satisfied and $\Psi^{(2)}$

[2] To derive this we first use the relation $\langle \phi, Q_B \Psi^{(2)} \rangle = -(-1)^\phi \langle Q_B \phi, \Psi^{(2)} \rangle$, where Q_B on the right-hand side is an integral of the BRST current j_B over a contour that encircles the origin counterclockwise, with the operator j_B placed to the left of $f \circ \phi(0)$ in the correlator. Using the identification of the surface $\mathcal{W}_{1+t}$, the contour can be deformed to $-V^+_{2(1+t)} + V^+_0$. In the correlator, we move the BRST current from the left of $f \circ \phi(0)$ to the right of it. This cancels $(-1)^\phi$, and the additional minus sign is canceled by reversing the orientation of the contour.

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itself is finite. Note that $V(0)\,V(t)$ in the limit $t \to 0$ can be finite or can be vanishing. We construct $\Psi^{(n)}$ for marginal operators with regular operator products in the next section. When the operator product of V with itself is singular, the formal solution $\Psi^{(2)}$ is not well defined. We discuss this case in section 4.

3. Solutions for marginal operators with regular operator products

In the previous section we constructed a well-defined solution to the equation $Q_B \Psi^{(2)} + \Psi^{(1)} * \Psi^{(1)} = 0$ when V has a regular operator product. In this section we generalize it to $\Psi^{(n)}$ for any n. We then present the solution that corresponds to the decay of an unstable D-brane in section 3.2. In section 3.3 we study marginal deformations in the lightcone direction and discuss the application to the solution that represents a string ending on a D-brane.

3.1 Solution

Once we understand how $\Psi^{(2)}$ in the form of (2.32) satisfies the equation of motion, it is easy to construct $\Psi^{(n)}$ satisfying $Q_B \Psi^{(n)} = \Phi^{(n)}$. It is given by

$$\begin{aligned} \langle\, \phi, \Psi^{(n)} \,\rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \,\langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, cV(1+t_1+t_2) \,\ldots \\ \times \mathcal{B}\, cV(1+t_1+t_2+\cdots+t_{n-1}) \,\rangle_{\mathcal{W}_{1+t_1+t_2+\cdots+t_{n-1}}} \,. \end{aligned} \tag{3.1}$$

Introducing the length parameters

$$\ell_i \equiv \sum_{k=1}^{i} t_k \,, \tag{3.2}$$

the solution can be written more compactly as

$$\boxed{\langle\, \phi, \Psi^{(n)} \,\rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \Big\langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \,\Big] \,\Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}} \,.} \tag{3.3}$$

See figure 1. The solution obeys the Schnabl gauge condition. It is remarkably simple contrasted with the expression one would obtain in Siegel gauge.

Let us now prove that the equation of motion is satisfied for (3.3). It is straightforward to generalize the calculation of $\langle\, \phi, Q_B \Psi^{(2)} \,\rangle$ in the previous section to that of $\langle\, \phi, Q_B \Psi^{(n)} \,\rangle$. The BRST operator in $\langle\, \phi, Q_B \Psi^{(n)} \,\rangle$ can be represented as an integral of the BRST current on $V^+_{2(1+\ell_{n-1})} - V^+_0$. Since cV is BRST closed, the BRST operator acts only on the insertions of $\mathcal{B}$'s:

$$\begin{aligned} \langle\, \phi, Q_B \Psi^{(n)} \,\rangle = -\sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \Big\langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{j-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \,\Big] \\ \times \mathcal{L}\, cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_k) \,\Big] \,\Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}} \,. \end{aligned} \tag{3.4}$$

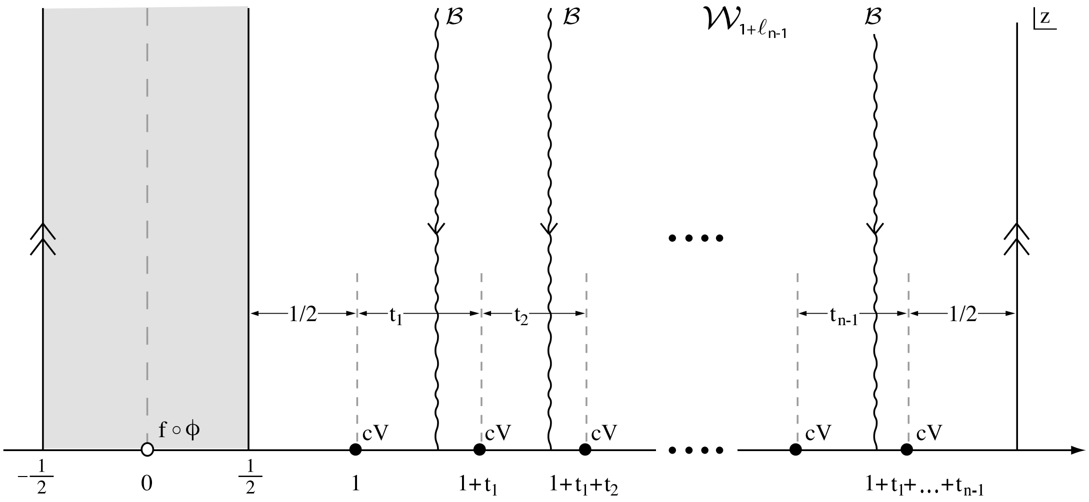

Figure 1: The surface $\mathcal{W}_{1+\ell_{n-1}}$ with the operator insertions used to construct the solution $\Psi^{(n)}$ given in (3.3). The parameters $t_1, t_2, \ldots, t_{n-1}$ must all be integrated from zero to one. The leftmost and rightmost vertical lines with double arrows are identified.

An insertion of $\mathcal{L}$ between $cV(1+\ell_{j-1})$ and $cV(1+\ell_j)$ corresponds to taking a derivative with respect to t_j. When operator products of V are regular, we have

$$\begin{aligned}
\langle\, \phi, Q_B \Psi^{(n)} \,\rangle &= -\sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1}\, \partial_{t_j} \Big\langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{j-1} \Big[\mathcal{B}\, cV(1+\ell_i) \Big] \\
&\qquad\qquad \times cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\mathcal{B}\, cV(1+\ell_k) \Big] \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}} \\
&= -\sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{j-1} \int_0^1 dt_{j+1} \ldots \int_0^1 dt_{n-1} \Big\langle\, f \circ \phi(0)\, cV(1) \\
&\qquad \times \prod_{i=1}^{j-1} \Big[\mathcal{B}\, cV(1+\ell_i) \Big]\, cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\mathcal{B}\, cV(1+\ell_k) \Big] \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}} \bigg|_{t_j=1} \\
&= -\sum_{j=1}^{n-1} \langle\, \phi, \Psi^{(j)} * \Psi^{(n-j)} \,\rangle \,.
\end{aligned} \tag{3.5}$$

The equation of motion is thus satisfied.[3]

We can also derive this expression of $\Psi^{(n)}$ by acting with B/L on $\Phi^{(n)}$. It is in fact interesting to see how the region of the integrals over $t_1, t_2, \ldots, t_{n-1}$ is reproduced. Let us demonstrate it taking the case of $\Psi^{(3)}$ as an example. Using the Schwinger representation

[3]We assume that operator products of more than two V's are also regular in order for the surface term from $t_j = 0$ to vanish. This additional regularity condition was overlooked in the first version of the paper on arXiv.

of B/L, the expression (2.26) for $\Psi^{(2)}$, and the identities (2.15) and (2.14), we have

$$
\begin{aligned}
\Psi^{(3)} &= -\int_0^\infty dT_2\, B\, e^{-T_2 L}\, \Big[\, \Psi^{(1)} * \Psi^{(2)} + \Psi^{(2)} * \Psi^{(1)} \,\Big] \\
&= -\int_0^\infty dT_2 \int_0^\infty dT_1\, B\, e^{-T_2 L} \bigg[\, \Psi^{(1)} * \Psi^{(1)} * (B - B_L^+)\, e^{-T_1(L-L_L^+)}\, \Psi^{(1)} \\
&\qquad\qquad + \Psi^{(1)} * (B - B_L^+)\, e^{-T_1(L-L_L^+)}\, \Psi^{(1)} * \Psi^{(1)} \bigg] \\
&= \int_0^\infty dT_1 \int_0^\infty dT_2 \bigg[\, \Psi^{(1)} * (B - B_L^+)\, e^{-T_2(L-L_L^+)}\, \Psi^{(1)} * (B - B_L^+)\, e^{-(T_1+T_2)(L-L_L^+)}\, \Psi^{(1)} \\
&\qquad + \Psi^{(1)} * (B - B_L^+) e^{-(T_1+T_2)(L-L_L^+)} \Psi^{(1)} * (B - B_L^+) e^{-T_2(L-L_L^+)} \Psi^{(1)} \bigg].
\end{aligned}
\tag{3.6}
$$

By changing variables as $\tau_1 = T_2$ and $\tau_2 = T_1 + T_2$ for the first term and as $\tau_2 = T_2$ and $\tau_1 = T_1 + T_2$ for the second term, the two terms combine into

$$
\Psi^{(3)} = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, \Psi^{(1)} * (B - B_L^+)\, e^{-\tau_1(L-L_L^+)}\, \Psi^{(1)} * (B - B_L^+)\, e^{-\tau_2(L-L_L^+)}\, \Psi^{(1)}\,. \tag{3.7}
$$

The same manipulations we performed with $\Psi^{(2)}$ give

$$
\Psi^{(3)} = \int_0^1 dt_1 \int_0^1 dt_2\, \Psi^{(1)} * e^{-(t_1-1)L_L^+} (-B_L^+)\, \Psi^{(1)} * e^{-(t_2-1)L_L^+} (-B_L^+)\, \Psi^{(1)} \tag{3.8}
$$

and the following expression in the CFT formulation:

$$
\langle\, \phi, \Psi^{(3)} \,\rangle = \int_0^1 dt_1 \int_0^1 dt_2\, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, cV(1+t_1+t_2)\, \rangle_{\mathcal{W}_{1+t_1+t_2}} \tag{3.9}
$$

in agreement with (3.3). It is not difficult to use induction to prove that for all n (3.3) follows from the action of B/L on $\Phi^{(n)}$.

We conclude the subsection by writing other forms of the solution that are suitable for explicit calculations. We represent the surface $\mathcal{W}_\alpha$ as the region between V_2^- and $V_{2(\alpha-1)}^+$. The operator $cV(1+\ell_{n-1})$ in (3.3) is then mapped to $cV(-1)$. We further transform $\langle\, \phi, \Psi^{(n+1)} \,\rangle$ in the following way:

$$
\begin{aligned}
\langle\, \phi, \Psi^{(n+1)} \,\rangle &= \int_0^1 dt_1 \ldots \int_0^1 dt_n \left\langle cV(-1)\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{n-1} \Big[\mathcal{B}\, cV(1+\ell_i) \Big] \mathcal{B} \right\rangle_{\mathcal{W}_{1+\ell_n}} \\
&= \int_0^1 dt_1 \ldots \int_0^1 dt_n \left\langle cV(-1)\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{n-1} \Big[V(1+\ell_i) \Big] \mathcal{B} \right\rangle_{\mathcal{W}_{1+\ell_n}} \\
&= -\int_0^1 dt_1 \ldots \int_0^1 dt_n\, \frac{1}{2+\ell_n} \\
&\quad \times \left\langle \int_{V_{2\ell_n}^+ - V_2^-} \frac{dz}{2\pi i}\, z\, b(z) \Big[cV(-1)\, f \circ \phi(0)\, cV(1) \Big] \prod_{i=1}^{n-1} \Big[V(1+\ell_i) \Big] \right\rangle_{\mathcal{W}_{1+\ell_n}}.
\end{aligned}
\tag{3.10}
$$

First we recursively used the relation $\mathcal{B}\, c(z)\, \mathcal{B} = \mathcal{B}$, which follows from $\{\, \mathcal{B},\, c(z)\,\} = 1$ and $\mathcal{B}^2 = 0$. In the last step, we used the identity

$$\int_{V^+_{2(\alpha-1)} - V^-_2} \frac{dz}{2\pi i}\, z\, b(z) = (\alpha+1) \int_{V^+_{2(\alpha-1)}} \frac{dz}{2\pi i}\, b(z) \quad \text{on} \quad \mathcal{W}_\alpha\,. \tag{3.11}$$

This follows from

$$\int_{V^-_2} \frac{dz_-}{2\pi i}\, z_-\, b(z_-) = \int_{V^+_{2(\alpha-1)}} \frac{dz_+}{2\pi i} \Big\{\, z_+ - (\alpha+1)\,\Big\}\, b(z_+) \quad \text{on} \quad \mathcal{W}_\alpha\,, \tag{3.12}$$

where the coordinate z_- for V^-_2 and the coordinate z_+ for $V^+_{2(\alpha-1)}$ are identified by $z_+ = z_- + \alpha + 1$. The contour $V^+_{2\ell_n} - V^-_2$ can be deformed to encircle $cV(-1)$, $f \circ \phi(0)$, and $cV(1)$, and we obtain

$$\begin{aligned} \langle\, \phi, \Psi^{(n+1)}\,\rangle = \int_0^1 dt_1 \ldots \int_0^1 dt_n\, \frac{1}{2+\ell_n} \Big\langle\, \Big\{ V(-1)\, f\circ\phi(0)\, cV(1) + cV(-1)\, f\circ\phi(0)\, V(1) \\ + cV(-1) \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, f\!\circ\!\phi(0) \right] cV(1) \Big\} \prod_{i=1}^{n-1} V(1+\ell_i) \Big\rangle_{\mathcal{W}_{1+\ell_n}}, \end{aligned} \tag{3.13}$$

where the contour in the last line encircles the origin counterclockwise.

When $\phi(0)$ factorizes into a matter part $\phi_m(0)$ and a ghost part $\phi_g(0)$, we can use the matter-ghost factorization of the correlator to give an alternative form of (3.3):

$$\begin{aligned} \langle\, \phi, \Psi^{(n)}\,\rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \Big\langle\, f\circ\phi_m(0) \prod_{i=0}^{n-1} V(1+\ell_i) \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}},\, m} \\ \times \Big\langle\, f\circ\phi_g(0)\, c(1)\, \mathcal{B}\, c(1+\ell_{n-1}) \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}},\, g}, \end{aligned} \tag{3.14}$$

where $\ell_0 \equiv 0$ and we denoted matter and ghost correlators by subscripts m and g, respectively. The ghost correlator in the above expression is $\langle\, \phi_g, \psi_{\ell_{n-1}}\,\rangle$ in (2.38). The algorithm for its calculation has been developed in [4, 8].

3.2 Rolling tachyon marginal deformation to all orders

We can now apply the general solution (3.13) to the special case of a marginal deformation corresponding to a rolling tachyon. For this purpose we pick the operator

$$V(z,\bar z) = e^{\frac{1}{\sqrt{\alpha'}} X^0(z,\bar z)} \tag{3.15}$$

restricted to the boundary $z = \bar z = y$ of the upper-half plane $\mathbb{H}$, where we write it as[4]

$$V(y) = e^{\frac{1}{\sqrt{\alpha'}} X^0(y)}\,, \qquad X^0(y) \equiv X^0(y,y)\,. \tag{3.16}$$

[4]We use the signature $(-,+,+,\ldots,+)$. For a point $z = \bar z = y$ on the boundary of $\mathbb{H}$ we write $X^\mu(y) \equiv X^\mu(y,y)$. The singular part of $X^\mu(y)X^\nu(y')$ is given by $-2\alpha'\eta^{\mu\nu}\ln|y-y'|$, and the mode expansion for a Neumann coordinate reads $i\partial_y X^\mu(y) = \sqrt{2\alpha'}\sum_m \frac{\alpha^\mu_m}{y^{m+1}}$. The basic correlator is $\langle e^{ik\cdot X(y)} e^{ik'\cdot X(y')}\rangle = (2\pi)^D \delta^{(D)}(k+k')|y-y'|^{2\alpha' k\cdot k'}$, where D is the spacetime dimension. The operator $e^{ik\cdot X(y)}$ has dimension $\alpha' k^2$ and transforms as $f\circ e^{ik\cdot X(y)} = |\frac{df}{dy}|^{\alpha' k^2} e^{ik\cdot X(f(y))}$. We do not use the doubling trick for the matter sector in section 3.2 and section 3.3. In these subsections, $\partial X^\mu \equiv \partial_z X^\mu + \partial_{\bar z} X^\mu$ when μ is a direction along the D-brane and $\partial X^\mu \equiv \partial_z X^\mu - \partial_{\bar z} X^\mu$ when μ is a direction transverse to the D-brane.

The operator $e^{ik\cdot X(y)}$ has dimension $\alpha' k^2$ and we can write

$$V(y) = e^{ik\cdot X(y)} \quad \text{with} \quad k^\mu = \frac{i}{\sqrt{\alpha'}}\big(1, \vec{0}\,\big) \;\rightarrow\; \alpha' k^2 = 1\,, \tag{3.17}$$

showing that V is a matter primary field of dimension one. We also have

$$V(y)V(0) \sim\, |y|^2 V(0)^2\,, \tag{3.18}$$

and the matter operator satisfies the requisite regularity condition.

We will also use exponential operators of X^0 with different exponents. We thus record the following transformation law and ordering results:

$$\begin{aligned} f\circ e^{\frac{1}{\sqrt{\alpha'}}\,nX^0(y)} &= \Big|\frac{df}{dy}\Big|^{n^2} e^{\frac{1}{\sqrt{\alpha'}}\,nX^0(f(y))}\,, &(3.19)\\ e^{\frac{1}{\sqrt{\alpha'}}\,mX^0(y)} e^{\frac{1}{\sqrt{\alpha'}}\,nX^0(y')} &= |y-y'|^{2mn} : e^{\frac{1}{\sqrt{\alpha'}}\,mX^0(y)} e^{\frac{1}{\sqrt{\alpha'}}\,nX^0(y')} :\,. &(3.20)\end{aligned}$$

Physically, deformation by cV represents a rolling tachyon solution in which the state of the system at time $x^0 = -\infty$ is the perturbative vacuum. We set $\Psi^{(1)}$ to be

$$\Psi^{(1)} = e^{\frac{1}{\sqrt{\alpha'}}X^0(0)}\, c_1|0\rangle \tag{3.21}$$

and calculate $\Psi^{(n)}$ with $n \geq 2$ which, by momentum conservation, must take the form

$$\Psi^{(n)} = e^{\frac{1}{\sqrt{\alpha'}}\,nX^0(0)}\Big[\beta_n\, c_1|0\rangle + \cdots\Big]\,, \quad n \geq 2\,. \tag{3.22}$$

In the above we have separated out the tachyon component, and higher-level fields are indicated by dots. The profile of the tachyon field T is determined by the coefficients β_n that we aim to calculate:

$$T(x^0) = \lambda\, e^{\frac{1}{\sqrt{\alpha'}}x^0} + \sum_{n=2}^{\infty}\beta_n\,\lambda^n e^{\frac{1}{\sqrt{\alpha'}}\,nx^0}\,. \tag{3.23}$$

Since the solution (for every component field) depends on λ and x^0 only through the combination $\lambda e^{\frac{1}{\sqrt{\alpha'}}x^0}$, a scaling of λ can be absorbed by a shift of x^0. We can therefore focus on the case $\lambda = \mp 1$. The sign of λ makes a physical difference. In our conventions the tachyon vacuum lies at some $T < 0$, so $\lambda = -1$ corresponds to the tachyon rolling in the direction of the tachyon vacuum, which we are mostly interested in. For $\lambda = +1$ the tachyon begins to roll towards the unbounded region of the potential. After setting $\lambda = \mp 1$, we write

$$T(x^0) = \mp\, e^{\frac{1}{\sqrt{\alpha'}}x^0} + \sum_{n=2}^{\infty}(\mp 1)^n\beta_n\, e^{\frac{1}{\sqrt{\alpha'}}\,nx^0}\,. \tag{3.24}$$

In order to extract the coefficients β_n from the solution we introduce test states ϕ_n and their BPZ duals:

$$|\phi_n\rangle = e^{-\frac{1}{\sqrt{\alpha'}}\,nX^0(0)}\, c_0 c_1|0\rangle\,, \qquad \langle\phi_n| = \lim_{y\to\infty}\langle 0|c_{-1}c_0 e^{-\frac{1}{\sqrt{\alpha'}}\,nX^0(y)}\,\frac{1}{|y|^{2n^2}}\,. \tag{3.25}$$

The state ϕ_n has dimension n^2-1. Using (3.22) we find

$$\langle \phi_n, \Psi^{(n)} \rangle = \langle \phi_n | \Psi^{(n)} \rangle = \beta_n \cdot (\mathrm{vol})\,, \qquad \mathrm{vol} = (2\pi)^D \delta^{(D)}(0)\,. \tag{3.26}$$

The spacetime volume (vol) always factors out, so we will simply use vol$=1$ in the following. We now use (3.13) to write $\beta_{n+1} = \langle \phi_{n+1}, \Psi^{(n+1)} \rangle$ as

$$\begin{aligned}\beta_{n+1} = \int_0^1 dt_1 \ldots \int_0^1 dt_n \frac{1}{2+\ell_n} \Big\langle \Big\{ & e^{\frac{1}{\sqrt{\alpha'}}X^0(-1)}\, f\circ(\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}}X^0}(1) \\ & + c e^{\frac{1}{\sqrt{\alpha'}}X^0}(-1)\, f\circ(\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, e^{\frac{1}{\sqrt{\alpha'}}X^0(1)} \\ & + c e^{\frac{1}{\sqrt{\alpha'}}X^0}(-1)\, f\circ c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}}X^0}(1) \Big\} \prod_{i=1}^{n-1} e^{\frac{1}{\sqrt{\alpha'}}X^0(1+\ell_i)} \Big\rangle_{\mathcal{W}_{1+\ell_n}}.\end{aligned} \tag{3.27}$$

In the last term, due to the simple structure of ϕ_{n+1}, the antighost line integral acts as b_0 and simply removes the c_0 part of the state. We must now evaluate the correlator on the right-hand side.

This calculation requires the map from the surface $\mathcal{W}_{1+\ell_n}$ to the upper-half plane. We recall that the surface $\mathcal{W}_0$ of unit width can be mapped to the upper-half plane by the function

$$g(z) = \frac{1}{2}\tan(\pi z)\,. \tag{3.28}$$

Due to the periodicity $g(z+1) = g(z)$, this map works independent of the position of the surface $\mathcal{W}_0$ in the direction of the real axis. Consequently, we merely need to rescale $\mathcal{W}_{1+\ell_n}$ to $\mathcal{W}_0$ by $z \to \frac{z}{2+\ell_n}$ and then map it to the upper-half plane by $g(z)$. The overall conformal transformation on the test states is therefore the map h given by

$$h(\xi) \equiv g\Big(\frac{1}{2+\ell_n}\, f(\xi)\Big)\,. \tag{3.29}$$

All other vertex operators are mapped with $g\big(\frac{1}{2+\ell_n}\, z\big)$. It is therefore natural to define

$$g_i \equiv g\Big(\frac{1+\ell_i}{2+\ell_n}\Big)\,, \qquad g_i' \equiv g'\Big(\frac{1+\ell_i}{2+\ell_n}\Big)\,, \quad i = 0,1,\ldots,n\,, \quad \ell_0 \equiv 0\,. \tag{3.30}$$

With these abbreviations, the correlator on the upper-half plane reads

$$\begin{aligned}\beta_{n+1} = \int d^n t\, \frac{h'(0)^{(n+1)^2-1}}{2+\ell_n} \Big\langle \Big\{ & \frac{g_0'}{2+\ell_n}\Big(e^{\frac{1}{\sqrt{\alpha'}}X^0(-g_0)}\,(\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}}X^0}(g_0) \\ & + c e^{\frac{1}{\sqrt{\alpha'}}X^0}(-g_0)\,(\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, e^{\frac{1}{\sqrt{\alpha'}}X^0(g_0)}\Big) \\ & + c e^{\frac{1}{\sqrt{\alpha'}}X^0}(-g_0)\, c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}}X^0}(g_0) \Big\} \prod_{i=1}^{n-1} \frac{g_i'}{2+\ell_n} e^{\frac{1}{\sqrt{\alpha'}}X^0(g_i)} \Big\rangle_{\mathbb{H}},\end{aligned} \tag{3.31}$$

where $h'(0) = \frac{1}{2+\ell_n}$ and we have defined $\int d^n t \equiv \int_0^1 dt_1 \ldots \int_0^1 dt_n$. We can now factor this into matter and ghost correlators:

$$\beta_{n+1} = \int d^n t\,(2+\ell_n)^{-(n+1)^2} \Big\langle e^{\frac{1}{\sqrt{\alpha'}}X^0(-g_0)} e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0(0)} e^{\frac{1}{\sqrt{\alpha'}}X^0(g_0)} \prod_{i=1}^{n-1} \frac{g_i'}{2+\ell_n} e^{\frac{1}{\sqrt{\alpha'}}X^0(g_i)} \Big\rangle_m$$
$$\times \left\langle \frac{g_0'}{2+\ell_n}\Big((\partial c)c(0)\,c(g_0) + c(-g_0)\,(\partial c)c(0)\Big) + c(-g_0)\,c(0)\,c(g_0) \right\rangle_g . \qquad (3.32)$$

The ghost correlator can be evaluated using $\big\langle c(-z)c(0)c(z)\big\rangle_g = -2z^3$ and $\big\langle \partial c\, c(0)\, c(z)\big\rangle_g = z^2$. Using also $-g_0 = g_n$ and $g_0' = g_n'$, we find

$$\beta_{n+1} = 2\int d^n t\,(2+\ell_n)^{-n(n+3)} \Big(\frac{g_0'}{2+\ell_n} - g_0\Big) \frac{g_0^2}{{g_0'}^2} \prod_{i=0}^{n} \big[g_i'\big] \Big\langle e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0(0)} \prod_{i=0}^{n} e^{\frac{1}{\sqrt{\alpha'}}X(g_i)} \Big\rangle_m . \qquad (3.33)$$

Evaluating the matter correlator, we obtain our final result for the coefficients of the rolling tachyon solution:

$$\boxed{\beta_{n+1} = 2\int d^n t\,(2+\ell_n)^{-n(n+3)} \Big(\frac{g_0'}{2+\ell_n} - g_0\Big) \frac{g_0^2}{{g_0'}^2} \Big[\prod_{i=0}^{n} \frac{g_i'}{g_i^{2(n+1)}}\Big] \prod_{0\le i<j\le n} (g_i - g_j)^2 .} \qquad (3.34)$$

Another way to derive (3.34) is to use (3.14). The ghost correlator, which gives the tachyon coefficient of ψ_{ℓ_n}, has been calculated in [4, 8]:

$$\langle\, f\circ(\partial c)c(0)\,c(1)\,\mathcal{B}\,c(1+\ell_n)\,\rangle_{\mathcal{W}_{1+\ell_n},g} = \frac{2+\ell_n}{\pi}\left[1 - \frac{2+\ell_n}{2\pi}\sin\frac{2\pi}{2+\ell_n}\right]\sin^2\frac{\pi}{2+\ell_n}$$
$$= 2\,(2+\ell_n)\,\frac{g_0^2}{g_0'}\left(1 - \frac{(2+\ell_n)\,g_0}{g_0'}\right) . \qquad (3.35)$$

The calculation of the matter correlator is straightforward:

$$\left\langle f\circ e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0(0)} \prod_{i=0}^{n} e^{\frac{1}{\sqrt{\alpha'}}X^0(1+\ell_i)} \right\rangle_{\mathcal{W}_{1+\ell_n},m}$$
$$= \Big(\frac{2}{\pi}\Big)^{(n+1)^2} \left[\prod_{i=0}^{n} \frac{(2+\ell_n)^{-2(n+1)}}{\pi^{-2(n+1)}} \sin^{-2(n+1)} \frac{\pi(1+\ell_i)}{2+\ell_n}\right] \prod_{0\le i<j\le n} \frac{(2+\ell_n)^2}{\pi^2} \sin^2\frac{\pi(\ell_i-\ell_j)}{2+\ell_n}$$
$$= (2+\ell_n)^{-(n+1)(n+2)} \left[\prod_{i=0}^{n} \frac{g_i'}{g_i^{2(n+1)}}\right] \prod_{0\le i<j\le n} (g_i - g_j)^2 . \qquad (3.36)$$

It is easy to see that (3.34) is reproduced.

The integrand in (3.34) is manifestly positive since $g'(z) > 0$ and $\frac{g_0'}{2+\ell_n} - g_0 > 0$. It follows that all β_{n+1} coefficients are positive. For $n = 1$ we find

$$\beta_2 = 8\int_0^1 dt\, \frac{\frac{g_0'}{2+t} - g_0}{(2+t)^4 g_0^4} = 8\int_0^1 dt \left(\frac{2\cot\big(\frac{\pi}{2+t}\big)}{2+t}\right)^4 \left(\frac{\pi}{2(2+t)\cos^2\big(\frac{\pi}{2+t}\big)} - \frac{1}{2}\tan\Big(\frac{\pi}{2+t}\Big)\right) . \qquad (3.37)$$

Surprisingly, analytic evaluation of the integral is possible using *Mathematica*:

$$\beta_2 = \frac{64}{243\sqrt{3}}\,. \tag{3.38}$$

This coefficient is the same as that of the Siegel-gauge solution [23]. For $n = 2$ the final integral can be evaluated numerically:

$$\beta_3 = 8\int_0^1 dt_1 \int_0^1 dt_2\, \frac{\left(\frac{g_0'}{2+t_1+t_2} - g_0\right) g_1' \left(g_0^2 - g_1^2\right)^2}{(2+t_1+t_2)^{10} g_0^8 g_1^6} \simeq 2.14766 \cdot 10^{-3}\,. \tag{3.39}$$

The results for the first few β_n are summarized in table 1. The resulting tachyon profile (3.24) takes the form

$$\begin{aligned} T(x^0) = \mp\, e^{\frac{1}{\sqrt{\alpha'}}x^0} &+ \ 0.15206\ \ e^{\frac{1}{\sqrt{\alpha'}}2x^0} \quad \mp 2.148\cdot 10^{-3}\, e^{\frac{1}{\sqrt{\alpha'}}3x^0} \\ &+ 2.619\cdot 10^{-6}\, e^{\frac{1}{\sqrt{\alpha'}}4x^0} \ \ \mp 2.791\cdot 10^{-10}\, e^{\frac{1}{\sqrt{\alpha'}}5x^0} \\ &+ 2.801\cdot 10^{-15}\, e^{\frac{1}{\sqrt{\alpha'}}6x^0} \ \ \mp 2.729\cdot 10^{-21}\, e^{\frac{1}{\sqrt{\alpha'}}7x^0} + \ldots \end{aligned} \tag{3.40}$$

The top sign gives us the physical solution: the tachyon rolls towards the tachyon vacuum, overshoots it, and then begins to develop larger and larger oscillations. The coefficients in the solution decrease so rapidly that the series seems to be absolutely convergent for any value of $\frac{x^0}{\sqrt{\alpha'}}$. Indeed, the n-th term T_n in the above series appears to take the form

$$|T_n| \sim 2.7\cdot 10^{-\frac{1}{2}n(n-1)}\, e^{\frac{1}{\sqrt{\alpha'}}nx^0}\,. \tag{3.41}$$

One then finds that the ratio of consecutive coefficients is

$$\left|\frac{T_{n+1}}{T_n}\right| \sim 10^{-n} e^{\frac{1}{\sqrt{\alpha'}}x^0} \simeq e^{-2.303\,n} e^{\frac{1}{\sqrt{\alpha'}}x^0}\,. \tag{3.42}$$

For any value of $\frac{x^0}{\sqrt{\alpha'}}$ the ratio becomes smaller than one for sufficiently large n, suggesting absolute convergence. It would be useful to do analytic estimates of β_n using (3.34) to confirm the above speculation.

It is interesting to compare the results with those of the p-adic model [19]. The relevant solution is discussed in section 4.2.2 of that paper and has the same qualitative behavior as the solution presented here: the tachyon rolls towards the minimum, overshoots it, and then develops ever-growing oscillations. The solution is of the form

$$\phi(t) = 1 - \sum_{n=1}^{\infty} a_n\, e^{\sqrt{2}nt}\,, \quad a_1 = 1\,. \tag{3.43}$$

The coefficients a_n can be calculated exactly with a simple recursion and fall off very rapidly, but an analytic expression for their large n behavior is not known. A fit of the values of a_n for $n = 3, \ldots, 13$ gives $\ln a_n \simeq -0.1625 + 1.506\,n - 1.389\,n^2$. (A fit with an n^3 term returns a very small coefficient for this term.) The fit implies that the ratio of two consecutive terms in the solution is

$$\left|\frac{a_{n+1}}{a_n}\right| e^{\sqrt{2}\,t} \sim e^{-2.778\,n+0.117} e^{\sqrt{2}\,t} \simeq 1.125\cdot 16^{-n} e^{\sqrt{2}\,t}\,. \tag{3.44}$$

n	β_n
2	$\frac{64}{243\sqrt{3}} \approx 0.152059$
3	$2.14766 \cdot 10^{-3}$
4	$2.61925 \cdot 10^{-6}$
5	$2.79123 \cdot 10^{-10}$
6	$2.80109 \cdot 10^{-15}$
7	$2.72865 \cdot 10^{-21}$

Table 1: Numerical values of the rolling tachyon profile coefficients.

This result suggests that the p-adic rolling solution is also absolutely convergent.

A low-level solution of the string theory rolling tachyon in Siegel gauge was also obtained in [19], where significant similarities with the p-adic solution were noted. The higher-level Siegel gauge analysis of the rolling tachyon in [23] confirmed the earlier analysis and added much confidence to the validity of the oscillatory solution. We believe that the exact analytic solution presented here has settled the issue convincingly.

3.3 Lightcone-like deformations

Another simple example of a marginal operator with regular operator products is provided by the lightcone-like operator

$$V(z) = \frac{i}{\sqrt{2\alpha'}}\,\partial X^+\,, \tag{3.45}$$

as usual, inserted at $z = \bar{z} = y$. Here $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^1)$ is a lightcone coordinate. (We could have also chosen $X^- = \frac{1}{\sqrt{2}}(X^0 - X^1)$.) The OPE of V with itself is regular: $\lim_{z\to 0} V(z)V(0) = V(0)^2$. The operator is dimension one and cV is BRST closed. We can construct a solution using the above $V(z)$ and our general result (3.13). If we consider some Dp-brane with $p < D-1$, we can choose x^1 to be a direction normal to the brane and the above matter deformation corresponds to giving constant expectation values to the time component of the gauge field on the brane and to the scalar field on the brane that represents the position of the brane.

To make the analysis a bit more nontrivial we consider the discussion of Michishita [25] on the Callan-Maldacena solution [26] for a string ending on a brane in the framework of OSFT. We choose

$$V(y) = \int dk_i\, A(k_i)\,\frac{i}{\sqrt{2\alpha'}}\,\partial X^+ e^{ik_iX^i}(y)\,, \tag{3.46}$$

where X^i's are the spatial directions on the brane. This operator has regular operator products: the exponentials $e^{ik_iX^i}(y)$ give positive powers of distances since k_i is spacelike. The operator $c\partial X^+ e^{ik_iX^i}$, however, has dimension $\alpha' k^2$, so unless $k_i = 0$ it is not BRST closed and the expression in (3.13) does not provide a solution. But it is not too far from a solution: if one chooses $A(k) \sim 1/k^2$, the action of Q_B on cV gives a delta function in position space.

We thus take $\Psi_A^{(1)} = V(0)c_1|0\rangle$ and, following [25], take its failure to be annihilated by Q_B to define the source term $J^{(1)}$ that hopefully would arise independently in a complete theory: $Q_B\Psi_A^{(1)} = J^{(1)}$. We can then calculate $\Psi_A^{(2)}$ which satisfies $Q_B\Psi_A^{(2)} + \Psi_A^{(1)} * \Psi_A^{(1)} = J^{(2)}$ for some $J^{(2)}$. While $BJ^{(1)} \neq 0$, we demand $BJ^{(n)} = 0$ for $n \geq 2$ following the approach of [25] in the Siegel-gauge case. Acting with B on the above equation for $\Psi_A^{(2)}$, we find

$$L\Psi_A^{(2)} + B\left(\Psi_A^{(1)} * \Psi_A^{(1)}\right) = 0 \quad \to \quad \Psi_A^{(2)} = -\frac{B}{L}\left(\Psi_A^{(1)} * \Psi_A^{(1)}\right). \tag{3.47}$$

Acting with Q_B on the solution, one confirms that

$$Q_B\Psi_A^{(2)} = -\Psi_A^{(1)} * \Psi_A^{(1)} + \frac{B}{L}\left(J^{(1)} * \Psi_A^{(1)} - \Psi_A^{(1)} * J^{(1)}\right) \tag{3.48}$$

so that the source term $J^{(2)}$ is indeed annihilated by B.

In calculating $\Psi_A^{(2)}$ in (3.47) with $L\Psi_A^{(1)} \neq 0$, we need to generalize our results in section 2.3 and find the action of B/L on a string field product $\chi * \chi'$ where χ and χ' are not annihilated by L but instead satisfy

$$B\chi = B\chi' = 0\,, \qquad L\chi = l_\chi\,\chi\,, \qquad L\chi' = l_{\chi'}\,\chi'\,. \tag{3.49}$$

The steps leading to (2.29) can be carried out analogously for this case with extra factors depending on l_χ and $l_{\chi'}$:

$$\frac{B}{L}(\chi * \chi') = (-1)^\chi \int_0^1 dt\, t^{(l_\chi + l_{\chi'})}\, \chi * e^{-(t-1)L_L^+}\,(-B_L^+)\,\chi'\,. \tag{3.50}$$

To construct $\Psi_A^{(2)}$, we need to express states of the type $\frac{B}{L}(\chi * \chi')$ as CFT correlators. As χ and χ' are primary fields of nonvanishing dimension, there are extra conformal factors in the sliver-frame expression for these states. Defining a shift function $s_l(z) = z + l$, we can express the generalization of (2.32) that accounts for these extra factors as

$$\begin{aligned}\langle\, \phi, \frac{B}{L}\left(\chi * \chi'\right)\rangle &= (-1)^\chi \int_0^1 dt\, t^{(l_\chi + l_{\chi'})}\,\langle\, f\circ\phi(0)\;\; s_1\circ f\circ\chi(0)\;\; \mathcal{B}\;\; s_{1+t}\circ f\circ\chi'(0)\,\rangle_{\mathcal{W}_{1+t}} \\ &= (-1)^\chi \int_0^1 dt\, \left(tf'(0)\right)^{(l_\chi + l_{\chi'})}\,\langle\, f\circ\phi(0)\;\chi(1)\;\mathcal{B}\;\chi'(1+t)\,\rangle_{\mathcal{W}_{1+t}}\,.\end{aligned} \tag{3.51}$$

Here we have explicitly carried out the conformal maps of χ and χ' to the sliver frame and used $s_l'(z) = 1$. It is now straightforward to carry out the construction of $\Psi_A^{(2)}$ by generalizing (3.13). This yields

$$
\begin{aligned}
\langle \phi, \Psi_A^{(2)} \rangle = \int dk_i dk_i' A(k_i) A(k_i') \\
\times \int_0^1 dt \, \frac{-\big(tf'(0)\big)^{\alpha'(k^2+k'^2)}}{(2+t)2\alpha'} \Big\langle \, \Big\{ \partial X^+ e^{ik_iX^i}(-1) \, f\circ\phi(0) \, c\partial X^+ e^{ik_i'X^i}(1) \\
+ c\partial X^+ e^{ik_iX^i}(-1) \left[\oint \frac{dz}{2\pi i} z b(z) f\circ\phi(0) \right] c\partial X^+ e^{ik_i'X^i}(1) \\
+ c\partial X^+ e^{ik_i'X^i}(-1) \, f\circ\phi(0) \, \partial X^+ e^{ik_iX^i}(1) \Big\} \Big\rangle_{\mathcal{W}_{1+t}} .
\end{aligned}
\tag{3.52}
$$

To obtain a Fock-space expression of $\Psi_A^{(2)}$, we follow the same steps leading to (5.50) of [8]. The map we need to perform on the correlator is $I \circ g$, so the total map on the test state ϕ is $I \circ h$. Here we have used g and h defined in (3.28) and (3.29), and $I(z) = -\frac{1}{z}$. Let us further define

$$
\hat{B} = \oint \frac{dz}{2\pi i} \, \frac{g^{-1}(z)}{(g^{-1})'(z)} b(z). \tag{3.53}
$$

Then we can start by mapping the correlator to the upper-half plane through g. Again, we will suppress all arguments of g and abbreviate

$$
g \equiv g\Big(\frac{1}{2+t}\Big) = -g\Big(-\frac{1}{2+t}\Big), \qquad g' \equiv g'\Big(\frac{1}{2+t}\Big) = g'\Big(-\frac{1}{2+t}\Big). \tag{3.54}
$$

We find

$$
\begin{aligned}
\langle \phi, \Psi_A^{(2)} \rangle = \int dk_i dk_i' A(k_i) A(k_i') \int_0^1 dt \, \frac{-1}{(2+t)2\alpha'} \Big(\frac{tf'(0)g'}{2+t} \Big)^{\alpha'(k^2+k'^2)} \\
\times \Big\langle \frac{g'}{2+t} \Big\{ \partial X^+ e^{ik_iX^i}(-g) \, h\circ\phi(0) \, c\partial X^+ e^{ik_i'X^i}(g) \\
+ c\partial X^+ e^{ik_i'X^i}(-g) \, h\circ\phi(0) \, \partial X^+ e^{ik_iX^i}(g) \Big\} \\
+ c\partial X^+ e^{ik_iX^i}(-g) \left[\hat{B} \, h\circ\phi(0) \right] c\partial X^+ e^{ik_i'X^i}(g) \Big\rangle_{\mathbb{H}} .
\end{aligned}
\tag{3.55}
$$

Here we used the fact that the operator $c\partial X^+ e^{ik_iX^i}$ has conformal dimension $\alpha' k^2$. We notice that the two terms in parenthesis can be transformed into each other through the map $g \to -g$. Therefore, we can drop one of them and simply take the g-even part of the other. We can now perform the remaining transformation with I to obtain an operator expression for $\Psi_A^{(2)}$:

$$\begin{aligned}\Psi_A^{(2)} &= \int dk_i dk'_i\, A(k_i)A(k'_i) \int_0^1 dt\, \frac{-1}{(2+t)2\alpha'} \Big(\frac{tf'(0)g'}{(2+t)g^2}\Big)^{\alpha'(k^2+k'^2)} \\ &\quad \times U_h^\star \bigg[\bigg\{ \frac{2g'}{(2+t)g^2}\, \partial X^+ e^{ik_iX^i}\big(-1/g\big)\, c\partial X^+ e^{ik'_iX^i}\big(1/g\big) \bigg\}_{g\text{-even}} \\ &\qquad + \hat{B}^\star\, c\partial X^+ e^{ik_iX^i}\big(-1/g\big)\, c\partial X^+ e^{ik'_iX^i}\big(1/g\big) \bigg] |0\rangle \\ &\equiv \int dk_i dk'_i\, A(k_i)A(k'_i)\, \Psi^{(2)}_{k,k'}\,. \end{aligned} \tag{3.56}$$

We would now like to determine the level expansion of $\Psi_A^{(2)}$, or equivalently, of its momentum decomposition $\Psi^{(2)}_{k,k'}$. We can either attempt a direct level expansion of the operator result (3.56) or use the test state formalism that we carried out in section 3.2. It is straightforward to carry out the first method for the case of vanishing momentum $k = k' = 0$, so we will start with this approach. We will then use the test state method to find the level expansion with full momentum dependence.

Let us start by the level expansion of $\Psi^{(2)}_{k,k'}$ in (3.56). We use the results in section 6.1 of [8] to obtain the following useful expansions:

$$\hat{B}^\star = b_0 + \frac{8}{3} b_{-2} + \dots\,, \qquad\qquad U_h^\star = (2+t)^{-L_0} + \dots \tag{3.57}$$

Here the dots denote higher-level corrections. We notice that self-contractions of ∂X^+ vanish as $\eta^{++} = 0$. We end up with the following mode expansions for the matter and ghost fields:

$$\begin{gathered} -\frac{1}{2\alpha'}\, \partial X^+(-1/g)\partial X^+(1/g)|0\rangle = \sum_{i<0,j<0} (-1)^{i+1}(\alpha_i^+\alpha_j^+) g^{i+j+2}|0\rangle\,, \\ c(\pm 1/g) = \sum_{m=-\infty}^{\infty} c_m \big(\pm g\big)^{m-1}\,, \quad \partial c(\pm 1/g) = -\sum_{m=-\infty}^{\infty} (m-1)c_m \big(\pm g\big)^m\,. \end{gathered} \tag{3.58}$$

The leading term in the level expansion of $\Psi^{(2)}_{(k,k')}$ in (3.56) for $k = k' = 0$ is given by

$$\begin{aligned} &\int_0^1 dt (2+t)^{-L_0-1} \bigg[\frac{2g'}{(2+t)g^2} (\alpha_{-1}^+)^2 c_1 - b_0 (\alpha_{-1}^+)^2 \frac{2}{g} c_0 c_1 \bigg] |0\rangle \\ &= 2\int_0^1 dt \frac{\frac{g'}{2+t} - g}{(2+t)^2 g^2} (\alpha_{-1}^+)^2 c_1 |0\rangle = \frac{4}{3\sqrt{3}} (\alpha_{-1}^+)^2 c_1 |0\rangle\,. \end{aligned} \tag{3.59}$$

The above component of the solution is exact to all orders in λ, as it cannot receive contributions from $\Psi^{(n)}$ with $n > 2$. The coefficient was determined analytically using *Mathematica*.

Let us now use the test state approach to determine this coefficient for general k and k'. In other words, we are trying to determine $\beta_{k,k'}$ in

$$\Psi^{(2)}_{k,k'} = \beta_{k,k'} e^{i(k_i+k'_i)X^i(0)} (\alpha_{-1}^+)^2 c_1 |0\rangle + \dots \tag{3.60}$$

As always, the dots denote higher-level contributions. The appropriate test state $\phi_{k,k'}$ such that $\langle \phi_{k,k'}, \Psi^{(2)}_{k,k'}\rangle = \beta_{k,k'} \cdot \text{(vol)}$ and its BPZ conjugate are given by

$$\begin{aligned}
|\phi_{k,k'}\rangle &= \frac{1}{2} e^{-i(k_i+k'_i)X^i(0)}\,(\alpha^-_{-1})^2 c_0 c_1|0\rangle = \frac{1}{2}\left(\frac{-1}{2\alpha'}\right)(\partial c)c\partial X^- \partial X^- e^{-i(k_i+k'_i)X^i}(0)\,|0\rangle\,,\\
\langle \phi_{k,k'}| &= \frac{1}{2}\lim_{y\to\infty}\langle 0|(\alpha^-_1)^2 c_{-1}c_0 e^{-i(k_i+k'_i)X^i(y)}\,\frac{1}{|y|^{2\alpha'(k+k')^2}}\,.
\end{aligned} \tag{3.61}$$

The state $\phi_{k,k'}$ has dimension $\alpha'(k+k')^2+1$. We can now evaluate $\beta_{k,k'}$ as in the calculation of β_{n+1} in section 3.2:

$$\begin{aligned}
\beta_{k,k'} &= \int_0^1 dt\,\frac{-1}{(2+t)2\alpha'}\left(\frac{tf'(0)g'}{2+t}\right)^{\alpha'(k^2+k'^2)}\\
&\quad\times\Big\langle \frac{g'}{2+t}\Big\{\partial X^+ e^{ik_iX^i}(-g) h\circ\phi_{k,k'}(0) c\partial X^+ e^{ik'_iX^i}(g)\\
&\qquad\qquad + c\partial X^+ e^{ik_iX^i}(-g) h\circ\phi_{k,k'}(0)\partial X^+ e^{ik'_iX^i}(g)\Big\}\\
&\qquad + c\partial X^+ e^{ik_iX^i}(-g)\left[\hat{B}\, h\circ\phi_{k,k'}(0)\right] c\partial X^+ e^{ik'_iX^i}(g)\Big\rangle_{\mathbb{H}}\\
&= \frac{1}{2}\left(\frac{1}{2\alpha'}\right)^2\int_0^1 dt\,\frac{h'(0)^{\alpha'(k+k')^2+1}}{2+t}\left(\frac{tf'(0)g'}{2+t}\right)^{\alpha'(k^2+k'^2)}\\
&\qquad\times\Big\langle \partial X^+ e^{ik_iX^i}(-g)\;\partial X^-\partial X^- e^{-i(k_i+k'_i)X^i}(0)\;\partial X^+ e^{ik'_iX^i}(g)\Big\rangle_m\\
&\qquad\times\Big\langle \frac{g'}{2+t}\;((\partial c)c(0)\,c(g) + c(-g)\,(\partial c)c(0)) + c(-g)\,c(0)\,c(g)\Big\rangle_g\,,
\end{aligned} \tag{3.62}$$

where we have again factored the correlator into the matter and ghost sectors. The matter contribution vanishes unless each ∂X^+ contracts with ∂X^-, and the ghost correlator has been calculated in section 3.2. We therefore have

$$\begin{aligned}
\beta_{k,k'} &= \left(\frac{1}{2\alpha'}\right)^2\int_0^1 dt\,\frac{h'(0)^{\alpha'(k+k')^2+1}}{2+t}\left(\frac{tf'(0)g'}{2+t}\right)^{\alpha'(k^2+k'^2)} 2\left(\frac{g'}{2+t}-g\right)g^2\\
&\qquad\times\Big\langle e^{ik_iX^i}(-g)\;e^{-i(k_i+k'_i)X^i}(0)\;e^{ik'_iX^i}(g)\Big\rangle_m\left(\frac{(2\alpha')\eta^{+-}}{g^2}\right)^2.
\end{aligned} \tag{3.63}$$

We evaluate the remaining matter correlator and use $h'(0)=\frac{1}{2+t}$ and $f'(0)=\frac{2}{\pi}$ to obtain

$$\beta_{k,k'} = 2\int_0^1 dt\,(2+t)^{-\alpha'(k+k')^2-2}\left(\frac{2tg'}{\pi(2+t)}\right)^{\alpha'(k^2+k'^2)}\left(\frac{g'}{2+t}-g\right)\frac{(2g)^{2\alpha' k\cdot k'}}{g^{2+2\alpha'(k+k')^2}}\,. \tag{3.64}$$

For general momenta the integral is complicated, but for $k=k'=0$ we recover the result from the operator expansion: $\beta_{k=0,k'=0}=\frac{4}{3\sqrt{3}}$. To summarize, our solution is

$$\begin{aligned}
\Psi &= \lambda\int dk_i\,A(k_i)e^{ik_iX^i(0)}\alpha^+_{-1}c_1|0\rangle\\
&\quad+\lambda^2\left(\int dk_idk'_i\,A(k_i)A(k'_i)\beta_{k,k'}e^{i(k_i+k'_i)X^i(0)}(\alpha^+_{-1})^2c_1|0\rangle+\dots\right)+O(\lambda^3)
\end{aligned} \tag{3.65}$$

with $\beta_{k,k'}$ given in (3.64).

4. Solutions for marginal operators with singular operator products

In the previous section, we constructed analytic solutions for marginal deformations when the operator V has regular operator products. In this section we generalize the construction to the case where V has the following singular OPE with itself:

$$V(z)\,V(w) \sim \frac{1}{(z-w)^2} + \text{regular}. \tag{4.1}$$

4.1 Construction of $\Psi^{(2)}$

The string field $\Psi^{(2)}$ in (2.32) is not well defined when V has the OPE (4.1). Let us define a regularized string field $\Psi_0^{(2)}$ as follows:

$$\langle\, \phi, \Psi_0^{(2)}\,\rangle = \int_{2\epsilon}^{1} dt\, \langle\, f\circ\phi(0)\, cV(1)\,\mathcal{B}\, cV(1+t)\,\rangle_{\mathcal{W}_{1+t}}\,. \tag{4.2}$$

The equation of motion is no longer satisfied by $\Psi_0^{(2)}$ because the surface term at $t=2\epsilon$ in (2.36) is nonvanishing. The BRST transformation of $\Psi_0^{(2)}$ is given by

$$\langle\, \phi, Q_B\Psi_0^{(2)}\,\rangle = -\,\langle\, \phi, \Psi^{(1)} * \Psi^{(1)}\,\rangle + \langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}}\,, \tag{4.3}$$

and we see that the second term on the right-hand side violates the equation of motion. Using the OPE

$$cV(-\epsilon)\, cV(\epsilon) = \frac{1}{2\epsilon}\, c\partial c(0) + O(\epsilon)\,, \tag{4.4}$$

the term violating the equation of motion can be written as

$$\langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} = \frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, c\partial c(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} + O(\epsilon)\,. \tag{4.5}$$

Since the operator $c\partial c$ is the BRST transformation of c, we recognize that the term (4.5) is BRST exact up to contributions which vanish as $\epsilon \to 0$. This crucial property makes it possible to satisfy the equation of motion by adding a counterterm to the regularized string field $\Psi_0^{(2)}$. We define the counterterm $\Psi_1^{(2)}$ by

$$\langle\, \phi, \Psi_1^{(2)}\,\rangle = -\,\frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, c(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}}\,. \tag{4.6}$$

The sum of $\Psi_0^{(2)}$ and $\Psi_1^{(2)}$ then solves the equation of motion in the limit $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\,\langle\, \phi, Q_B\,(\Psi_0^{(2)} + \Psi_1^{(2)}) + \Psi^{(1)} * \Psi^{(1)}\,\rangle = 0\,. \tag{4.7}$$

This is not yet the end of the story, as we must also require that the solution be finite as $\epsilon \to 0$. Since $\Psi^{(1)} * \Psi^{(1)}$ is a finite state, $Q_B\,(\Psi_0^{(2)} + \Psi_1^{(2)})$ is also finite in the limit $\epsilon \to 0$. This implies that while the state $\Psi_0^{(2)} + \Psi_1^{(2)}$ can be divergent, the divergent terms must be BRST closed. It follows that a finite solution can be obtained by simply subtracting the divergent terms from $\Psi_0^{(2)} + \Psi_1^{(2)}$. Let us isolate the divergent terms in $\Psi_0^{(2)}$. Using the anticommutation relation $\{\, \mathcal{B}, c(z)\,\} = 1$, the operator insertions in $\Psi_0^{(2)}$ can be written as

$$\begin{aligned} cV(1)\,\mathcal{B}\,cV(1+t) &= cV(1)\,V(1+t) - cV(1)\,cV(1+t)\,\mathcal{B} \\ &= \frac{1}{t^2}\,c(1) - \frac{1}{t}\,c\partial c(1)\,\mathcal{B} + O(t^0)\,. \end{aligned} \tag{4.8}$$

Using the formula

$$\begin{aligned} \langle\, \mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\,\ldots\,\mathcal{O}_n(z_n)\,\rangle_{\mathcal{W}_{\alpha+\delta\alpha}} &= \langle\, \mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\,\ldots\,\mathcal{O}_n(z_n)\,\rangle_{\mathcal{W}_\alpha} \\ &\quad + \delta\alpha\,\langle\, \mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\,\ldots\,\mathcal{O}_n(z_n)\,\mathcal{L}\,\rangle_{\mathcal{W}_\alpha} + O(\delta\alpha^2)\,, \end{aligned} \tag{4.9}$$

valid for any set of operators $\mathcal{O}_i$, we find

$$\begin{aligned} \langle f\circ\phi(0)\,cV(1)\,\mathcal{B}\,cV(1+t)\rangle_{\mathcal{W}_{1+t}} &= \frac{1}{t^2}\,\langle f\circ\phi(0)\,c(1)\rangle_{\mathcal{W}_1} \\ &\quad + \frac{1}{t}\,\langle f\circ\phi(0)\,[c(1)\mathcal{L} - c\partial c(1)\mathcal{B}]\rangle_{\mathcal{W}_1} + O(t^0) \\ &= \frac{1}{t^2}\,\langle\, f\circ\phi(0)\,c(1)\,\rangle_{\mathcal{W}_1} + \frac{1}{t}\,\langle\,\phi,\,\psi_0'\,\rangle + O(t^0)\,, \end{aligned} \tag{4.10}$$

where in the last equality we have used the expression for ψ_0' [5, 8]. The first term on the right-hand side is not BRST closed. After integration over t, it gives a divergent term of $O(1/\epsilon)$ which is precisely canceled by the divergent term from $\Psi_1^{(2)}$, as expected. The integral over t of the second term gives a divergent term of $O(\ln\epsilon)$ which is not canceled but, as expected, is BRST closed. (It is in fact BRST exact.) If we define the counterterm $\Psi_2^{(2)}$ by

$$\Psi_2^{(2)} = \ln(2\epsilon)\,\psi_0'\,, \tag{4.11}$$

we finally assemble a string field $\Psi^{(2)}$ that is finite and satisfies the equation of motion as follows:

$$\Psi^{(2)} = \lim_{\epsilon\to 0}\Big[\,\Psi_0^{(2)} + \Psi_1^{(2)} + \Psi_2^{(2)}\Big]\,. \tag{4.12}$$

We can also write the solution as

$$\Psi^{(2)} = \lim_{\epsilon\to 0}\left[\,\Psi_0^{(2)} - \frac{1}{\pi\epsilon}\,c_1|0\rangle + \ln(2\epsilon)\,\psi_0' + \frac{1}{\pi}\,L^+\,c_1|0\rangle\right], \tag{4.13}$$

using the following operator expression for $\Psi_1^{(2)}$:

$$\Psi_1^{(2)} = -\frac{1}{\pi\epsilon}\,e^{-\epsilon L^+}\,c_1|0\rangle = -\frac{1}{\pi\epsilon}\,c_1|0\rangle + \frac{1}{\pi}\,L^+\,c_1|0\rangle + O(\epsilon)\,. \tag{4.14}$$

Our construction of $\Psi^{(2)}$ did not rely on any property of V other than the OPE (4.1). The OPE (4.1) is more restrictive than the generic OPE of a dimension-one primary field. For example, we may have

$$V(z)\,V(w) \sim \frac{1}{(z-w)^2} + \frac{1}{z-w}\,U(w)\,, \tag{4.15}$$

where $U(w)$ is some matter primary field of dimension one. In this case, V would not be exactly marginal. Indeed, there must be a dimension-one primary field $\bar{U}$ such that $\langle\bar{U}(z)U(0)\rangle = 1/z^2$. The OPE (4.15) then implies that the three-point function $\langle V\,V\,\bar{U}\rangle$ is nonvanishing, while a necessary condition for the exact marginality of V is the vanishing of $\langle VVW\rangle$ for all dimension-one primary fields W. (See, for example, [27].) Thus we expect that our construction of $\Psi^{(2)}$ should not go through if the OPE takes the form (4.15). Let us see this explicitly. In this case (4.5) is replaced by

$$\begin{aligned}\langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} = {}& \frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, c\partial c(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} \\ &+ \langle\, f\circ\phi(0)\, c\partial cU(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} + O(\epsilon)\,.\end{aligned} \tag{4.16}$$

The second term on the right-hand side is finite in the limit $\epsilon\to 0$. The operator $c\partial cU$ is BRST closed, but it is *not* BRST exact. Therefore the equation of motion cannot be satisfied by adding a counterterm.

4.2 Gauge condition, L eigenstates, and divergence structure

All the terms of $\Psi^{(2)}$ in (4.13) are annihilated by B except $L^+c_1|0\rangle$:

$$BL^+c_1|0\rangle = [B, L^+]c_1|0\rangle = B^+c_1|0\rangle \neq 0\,. \tag{4.17}$$

Thus, rather curiously, $\Psi^{(2)}$ violates the Schnabl gauge condition. It appears that this violation is intrinsic. While we can add an arbitrary BRST closed state Z to $\Psi^{(2)}$, we believe that no choice of Z can restore the Schnabl gauge condition. Indeed, assume that such a state Z exists:

$$B(L^+c_1|0\rangle + Z) = 0\,, \quad Q_BZ = 0\,. \tag{4.18}$$

Acting with Q_B on this equation, we find that Z must satisfy

$$L\,Z = -Q_BBL^+c_1|0\rangle\,. \tag{4.19}$$

Note that while the left-hand side is in the image of L, the right-hand side is in the kernel of L because $[L, Q_B] = [L, B] = 0$ and $LL^+c_1|0\rangle = 0$. We believe that (4.19) has no solution for Z, though we do not have a proof.[5]

This obstruction in preserving the Schnabl gauge condition when V has the singular OPE (4.1) is rather unexpected. To gain some insight, let us reconsider the situation in Siegel gauge. In Siegel gauge the equations of motion (2.4) are solved by setting

$$\Psi^{(n)} = \frac{b_0}{L_0}\Phi^{(n)}\,. \tag{4.20}$$

It turns out that the right-hand side is well defined and thus manifestly obeys the gauge condition because $\Phi^{(n)}$ has no overlap with states in the kernel of L_0. When the equations of motion have a solution, $\Phi^{(n)}$ is a BRST-exact state of ghost number two. The only

[5] If an operator is diagonalizable, its kernel and its image have no nontrivial overlap. Since L is non-hermitian, it is not a priori clear if it can be diagonalized. In principle a state Z solving (4.19) may exist if L has a suitable Jordan structure, but we find this unlikely.

BRST-exact state of ghost number two in the kernel of L_0 is $Q_B c_0|0\rangle = 2c_1c_{-1}|0\rangle$. We are claiming that $\Phi^{(n)}$ has no overlap with $c_1c_{-1}|0\rangle$. This is shown using twist symmetry in the ghost sector. For a generic state in the Fock space

$$|\phi\rangle = \{\text{matter oscillators}\}\, b_{-m_j}\cdots b_{-m_1}c_{-n_k}\cdots c_{-n_1}|0\rangle\,, \qquad m_i \geq 2\,,\; n_i \geq -1\,, \tag{4.21}$$

the ghost-twist eigenvalue is defined to be

$$1 + \sum_{i=1}^{j} m_i + \sum_{i=1}^{k} n_i \quad (\text{mod } 2)\,. \tag{4.22}$$

The linearized solution $\Psi^{(1)}$ is even under ghost twist, which implies that $\Phi^{(2)} = -\Psi^{(1)}*\Psi^{(1)}$ is also even. On the other hand, the problematic state $c_1c_{-1}|0\rangle$ is odd. This shows that $\Phi^{(2)}$ has no overlap with it. A little inductive argument can be used to extend this result to $\Phi^{(n)}$ with $n > 2$. Assuming that all the states $\Psi^{(k)}$ with $k < n$ are even, we see that $\Phi^{(n)}$, which consists of symmetrized star products of the states $\Psi^{(k)}$ with $k < n$, is also even. Hence there is no obstruction in finding $\Psi^{(n)} = \frac{b_0}{L_0}\Phi^{(n)}$. The operator b_0/L_0 preserves twist, so $\Psi^{(n)}$ is even, and the induction can proceed to the next step.

We now perform a similar analysis for the case of Schnabl gauge. The formal solution

$$\Psi^{(n)} = \frac{B}{L}\Phi^{(n)} \tag{4.23}$$

is well defined if and only if $\Phi^{(n)}$ has no overlap with states in the kernel of L. While we do not have a complete understanding of the spectrum of L, we will find a consistent picture by assuming that $\Phi^{(n)}$ can be expanded in a sum of eigenstates of L with integer eigenvalues $L \geq -1$.[6] We can systematically enumerate the L eigenstates that have ghost number two and are BRST exact within a subspace of states which can appear in the expansion of $\Phi^{(n)}$. It will be sufficient to focus on states with $L \leq 0$. We believe that the only such states are as follows.

- $L = -1$: the state $c_1c_0|0\rangle = Q_B c_1|0\rangle$.
- $L = 0$: the state $c_1c_{-1}|0\rangle = \frac{1}{2}Q_B c_0|0\rangle$.
- $L = 0$: the state $L^+c_1c_0|0\rangle = Q_B L^+c_1|0\rangle$.

Contrasting the kernel of L with the kernel of L_0, we see the surprising appearance of the extra state $L^+c_1c_0|0\rangle$. Since this state is *even* under ghost twist, it can a priori appear in $\Phi^{(n)}$. The first state $c_1c_{-1}|0\rangle$ with $L = 0$ cannot appear, as we have argued before. We can write the following ansatz for a finite $\Phi^{(n)}$:

$$\Phi^{(n)} = \alpha^{(n)}c_1c_0|0\rangle + \beta^{(n)}L^+c_1c_0|0\rangle + \Phi^{(n)}_{>}\,, \tag{4.24}$$

where $\Phi^{(n)}_{>}$ only contains terms with positive eigenvalues of L. The most general $\Psi^{(n)}$ that satisfies the equation $Q_B\Psi^{(n)} = \Phi^{(n)}$ is the manifestly finite string field

$$\Psi^{(n)} = \alpha^{(n)}c_1|0\rangle + \beta^{(n)}L^+c_1|0\rangle + \frac{B}{L}\Phi^{(n)}_{>} + (\text{BRST closed})\,. \tag{4.25}$$

[6]Here and in what follows we use L to denote the eigenvalue of L as well.

If $\beta^{(n)} \neq 0$, the term $L^+ c_1|0\rangle$ violates the gauge condition. In the following we will not write the BRST-closed term that plays no role.

We are now going to establish a precise relationship between the violation of the gauge condition and the divergences that can arise in the Schwinger representation of the action of B/L when the matter operator has singular operator products. When B/L acts on $\Phi^{(n)}_{>}$, we can use its Schwinger representation

$$\frac{B}{L} = \lim_{\Lambda\to\infty} \int_0^\Lambda dt\, Be^{-tL} = \frac{B}{L} - \lim_{\Lambda\to\infty} e^{-\Lambda L}\frac{B}{L}\,, \tag{4.26}$$

since the boundary term vanishes in the limit. Thus we rewrite (4.25) as

$$\begin{aligned}\Psi^{(n)} &= \alpha^{(n)} c_1|0\rangle + \beta^{(n)} L^+ c_1|0\rangle + \lim_{\Lambda\to\infty}\int_0^\Lambda dt\, Be^{-tL}(\Phi^{(n)} - \alpha^{(n)} c_1 c_0|0\rangle - \beta^{(n)} L^+ c_1 c_0|0\rangle) \\ &= \lim_{\Lambda\to\infty}\left[\left(\int_0^\Lambda dt\, Be^{-tL}\Phi^{(n)}\right) + e^\Lambda \alpha^{(n)} c_1|0\rangle - \Lambda\beta^{(n)} BL^+ c_1 c_0|0\rangle\right] + \beta^{(n)} L^+ c_1|0\rangle.\end{aligned} \tag{4.27}$$

Note that we have

$$BL^+ c_1 c_0|0\rangle = \pi\psi_0'\,. \tag{4.28}$$

Since the string field $\Psi^{(n)}$ is finite, we see that

$$\int_0^\Lambda dt\, Be^{-tL}\Phi^{(n)} = -e^\Lambda \alpha^{(n)} c_1|0\rangle + \Lambda\,\pi\beta^{(n)}\,\psi_0' + \text{finite}\,. \tag{4.29}$$

We have thus learned that the divergences of the integral on the left-hand side, which performs the naive inversion of Q_B on $\Phi^{(n)}$, are directly related to the $L = -1$ and $L = 0$ eigenstates in the decomposition of $\Phi^{(n)}$. Moreover, the coefficient of the divergence of $O(\Lambda)$ is correlated with the coefficient of the Schnabl-gauge violating term $L^+ c_1|0\rangle$.

The divergences in (4.29) can only arise from the collision of the cV insertions on the boundary of the world-sheet. If V has regular operator products, all integrals are manifestly finite, $\alpha^{(n)} = \beta^{(n)} = 0$ for any n, $\Psi^{(n)}$ satisfies the Schnabl gauge condition, and the naive prescription $Q_B^{-1} = B/L$ is adequate to handle this case, as discussed in section 3. On the other hand, if V has a singular OPE with itself, (4.27) severely constrains the structure of the result. Let us look at the case of $\Psi^{(2)}$. To begin with, note that the integral

$$\int_0^\Lambda dt\, Be^{-tL}\Phi^{(2)} \tag{4.30}$$

is in fact the regularized $\Psi_0^{(2)}$ with the identification $\Lambda = -\ln(2\epsilon)$. Substituting this in (4.27), our general analysis predicts

$$\Psi^{(2)} = \lim_{\epsilon\to 0}\left[\Psi_0^{(2)} + \frac{\alpha^{(2)}}{2\epsilon}\, c_1|0\rangle + \ln(2\epsilon)\pi\,\beta^{(2)}\,\psi_0' + \beta^{(2)}\,L^+\,c_1|0\rangle\right] \tag{4.31}$$

in complete agreement with the explicit result (4.13) with $\alpha^{(2)} = -2/\pi$ and $\beta^{(2)} = 1/\pi$.

The analysis can be extended to $\Psi^{(n)}$ with $n > 2$. An interesting simplification occurs if $V = i\sqrt{\frac{2}{\alpha'}}\partial X$. Since the number of ∂X is conserved mod 2 under Wick contractions, the coefficients $\alpha^{(n)}$ and $\beta^{(n)}$ are zero for odd n. It follows that for odd n the integral (4.29) is finite. In particular, we expect that for $V = i\sqrt{\frac{2}{\alpha'}}\partial X$ the most general $\Psi^{(3)}$ is given by

$$\Psi^{(3)} = - \lim_{\Lambda\to\infty} \int_0^\Lambda dt\, B e^{-tL} \left(\Psi^{(1)} * \Psi^{(2)} + \Psi^{(2)} * \Psi^{(1)}\right) + (\text{BRST closed})\,, \qquad (4.32)$$

where the $\Lambda \to \infty$ limit is guaranteed to be finite.

While $\Psi^{(3)}$ may be obtained this way (setting the arbitrary BRST closed terms to zero and performing the integral by brute force), in the following subsection we will follow a route analogous to the one in section 4.1. We will start with a regularized $\Psi_0^{(3)}$ and systematically look for counterterms such that the final state $\Psi^{(3)}$ satisfies the equation of motion and is finite. The arguments in this section strongly suggest that a finite string field $\Psi^{(n)}$ satisfying the equation of motion exists for all n and it can be written as a regularized string field plus counterterms.

4.3 Construction of $\Psi^{(3)}$

In this subsection we perform an explicit construction of $\Psi^{(3)}$ for V with the OPE (4.1). The first step is to regularize (3.3) and define $\Psi_0^{(3)}$ by

$$\langle \phi, \Psi_0^{(3)} \rangle = \int_{2\epsilon}^1 dt_1 \int_{2\epsilon}^1 dt_2 \,\langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, cV(1+t_1+t_2)\,\rangle_{\mathcal{W}_{1+t_1+t_2}}\,. \quad (4.33)$$

The BRST transformation of $\Psi_0^{(3)}$ is given by

$$\langle \phi, Q_B \Psi_0^{(3)} \rangle = -\,\langle\, \phi, \Psi^{(1)} * \Psi_0^{(2)} + \Psi_0^{(2)} * \Psi^{(1)}\,\rangle + R_1 + R_2\,, \qquad (4.34)$$

where

$$\begin{aligned} R_1 &= \int_{2\epsilon}^1 dt_2 \,\langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\, \mathcal{B}\, cV(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}}\,, \\ R_2 &= \int_{2\epsilon}^1 dt_1 \,\langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, cV(1+t_1+2\epsilon)\,\rangle_{\mathcal{W}_{1+t_1+2\epsilon}}\,. \end{aligned} \qquad (4.35)$$

As in the case of $Q_B\Psi_0^{(2)}$, the contributions R_1 and R_2 from the surface terms at $t_1 = 2\epsilon$ and at $t_2 = 2\epsilon$, respectively, are nonvanishing. We also need to reproduce $-\Psi^{(1)} * \Psi_1^{(2)} - \Psi_1^{(2)} * \Psi^{(1)}$ and $-\Psi^{(1)} * \Psi_2^{(2)} - \Psi_2^{(2)} * \Psi^{(1)}$ to satisfy the equation of motion. It is not difficult to realize that the BRST transformation of $\Psi_1^{(3)}$ defined by

$$\Psi_1^{(3)} = -\int_{2\epsilon}^1 dt_1\, \Psi^{(1)} * B_L^+\, e^{(1-t_1)L_L^+}\, \Psi_1^{(2)} - \int_{2\epsilon}^1 dt_2\, \Psi_1^{(2)} * B_L^+\, e^{(1-t_2)L_L^+}\, \Psi^{(1)} \qquad (4.36)$$

cancels the divergent terms from the OPE's of $cV(1)\,cV(1+2\epsilon)$ in R_1 and of $cV(1+t_1)\,cV(1+t_1+2\epsilon)$ in R_2 and reproduces $-\Psi^{(1)} * \Psi_1^{(2)} - \Psi_1^{(2)} * \Psi^{(1)}$. We also introduce $\Psi_2^{(3)}$ defined by

$$\Psi_2^{(3)} = -\int_{2\epsilon}^1 dt_1\, \Psi^{(1)} * B_L^+\, e^{(1-t_1)L_L^+}\, \Psi_2^{(2)} - \int_{2\epsilon}^1 dt_2\, \Psi_2^{(2)} * B_L^+\, e^{(1-t_2)L_L^+}\, \Psi^{(1)} \qquad (4.37)$$

so that its BRST transformation reproduces $-\Psi^{(1)} * \Psi_2^{(2)} - \Psi_2^{(2)} * \Psi^{(1)}$.

However, this is not the whole story. First, when t_2 in R_1 is of $O(\epsilon)$, three V's are simultaneously close so that we cannot simply replace two of them by the most singular term of the OPE. The same remark applies to R_2 when t_1 is of $O(\epsilon)$. Secondly, while the contributions from the surface terms at $t_1 = 2\epsilon$ or at $t_2 = 2\epsilon$ in the calculation of $Q_B \Psi_2^{(3)}$ turn out to vanish in the limit $\epsilon \to 0$, the corresponding contributions in the calculation of $Q_B \Psi_1^{(3)}$ turn out to be *finite* and *not* BRST exact. These contributions have to be canceled in order for the equation of motion to be satisfied.

We thus need to calculate R_1, R_2, $Q_B \Psi_1^{(3)}$, and $Q_B \Psi_2^{(3)}$. The calculations of $Q_B \Psi_1^{(3)}$ and $Q_B \Psi_2^{(3)}$ are universal for any V which has the OPE (4.1), while those of R_1 and R_2 are not. Let us begin with $Q_B \Psi_1^{(3)}$. It is convenient to use the CFT description of $\Psi_1^{(3)}$ given by

$$\begin{aligned} \langle \phi, \Psi_1^{(3)} \rangle = & -\frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_1 \, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\, c(1+t_1+\epsilon)\, \rangle_{\mathcal{W}_{1+t_1+2\epsilon}} \\ & -\frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_2 \, \langle\, f \circ \phi(0)\, c(1+\epsilon)\, \mathcal{B}\, cV(1+t_2+2\epsilon)\, \rangle_{\mathcal{W}_{1+t_2+2\epsilon}} \,. \end{aligned} \tag{4.38}$$

The BRST transformation of $\Psi_1^{(3)}$ is

$$\langle \phi, Q_B \Psi_1^{(3)} \rangle = -\langle\, \phi, \Psi^{(1)} * \Psi_1^{(2)} + \Psi_1^{(2)} * \Psi^{(1)}\, \rangle + \widetilde{R}_1 + \widetilde{R}_2 + \widetilde{R}_3 \,, \tag{4.39}$$

where

$$\begin{aligned} \widetilde{R}_1 &= -\frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_2 \, \langle\, f \circ \phi(0)\, c\partial c(1+\epsilon)\, \mathcal{B}\, cV(1+t_2+2\epsilon)\, \rangle_{\mathcal{W}_{1+t_2+2\epsilon}} \,, \\ \widetilde{R}_2 &= -\frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_1 \, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\, c\partial c(1+t_1+\epsilon)\, \rangle_{\mathcal{W}_{1+t_1+2\epsilon}} \,, \\ \widetilde{R}_3 &= -\frac{1}{2\epsilon} \langle\, f \circ \phi(0)\, cV(1)\, c(1+3\epsilon)\, \rangle_{\mathcal{W}_{1+4\epsilon}} - \frac{1}{2\epsilon} \langle\, f \circ \phi(0)\, c(1+\epsilon)\, cV(1+4\epsilon)\, \rangle_{\mathcal{W}_{1+4\epsilon}} \,. \end{aligned} \tag{4.40}$$

As we mentioned earlier, the BRST transformation of $\Psi_1^{(3)}$ reproduces $-\Psi^{(1)} * \Psi_1^{(2)} - \Psi_1^{(2)} * \Psi^{(1)}$, and $\widetilde{R}_1$ and $\widetilde{R}_2$ cancel part of R_1 and R_2, respectively. The last term $\widetilde{R}_3$ is finite in the limit $\epsilon \to 0$ and not BRST exact:

$$\widetilde{R}_3 = -3\, \langle\, f \circ \phi(0)\, c\partial cV(1)\, \rangle_{\mathcal{W}_1} + O(\epsilon) \,. \tag{4.41}$$

Let us next calculate $Q_B \Psi_2^{(3)}$. It is again convenient to use the CFT description of $\Psi_2^{(3)}$:

$$\begin{aligned} \langle\, \phi, \Psi_2^{(3)}\, \rangle = & \ \ln(2\epsilon) \int_{2\epsilon}^{1} dt_1 \, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\ Q_B \cdot [\, \mathcal{B}\, c(1+t_1)\,]\, \rangle_{\mathcal{W}_{1+t_1}} \\ & + \ln(2\epsilon) \int_{2\epsilon}^{1} dt_2 \, \langle\, f \circ \phi(0)\ Q_B \cdot [\, \mathcal{B}\, c(1)\,]\ \mathcal{B}\, cV(1+t_2)\, \rangle_{\mathcal{W}_{1+t_2}} \,. \end{aligned} \tag{4.42}$$

The BRST transformation of $\Psi_2^{(3)}$ is given by

$$\begin{aligned} \langle\, \phi, Q_B \Psi_2^{(3)}\, \rangle = & -\langle\, \phi,\ \Psi^{(1)} * \Psi_2^{(2)} + \Psi_2^{(2)} * \Psi^{(1)}\, \rangle \\ & - \ln(2\epsilon)\, \langle\, f \circ \phi(0)\ Q_B \cdot [\, cV(1)\, \mathcal{B}\, c(1+2\epsilon)\,]\, \rangle_{\mathcal{W}_{1+2\epsilon}} \\ & + \ln(2\epsilon)\, \langle\, f \circ \phi(0)\ Q_B \cdot [\, \mathcal{B}\, c(1)\, cV(1+2\epsilon)\,]\, \rangle_{\mathcal{W}_{1+2\epsilon}} \,. \end{aligned} \tag{4.43}$$

Since the BRST transformations of $cV(1)\,\mathcal{B}\,c(1+2\epsilon)$ and $\mathcal{B}\,c(1)\,cV(1+2\epsilon)$ are both of $O(\epsilon)$, the last two terms vanish in the limit $\epsilon \to 0$. We have thus shown that

$$\lim_{\epsilon\to 0} \langle\, \phi, Q_B \Psi_2^{(3)} + \Psi^{(1)} * \Psi_2^{(2)} + \Psi_2^{(2)} * \Psi^{(1)} \,\rangle = 0\,. \tag{4.44}$$

To summarize, we have seen that the BRST transformation of $\Psi_0^{(3)} + \Psi_1^{(3)} + \Psi_2^{(3)}$ reproduces $-\Psi^{(1)} * \Psi^{(2)} - \Psi^{(2)} * \Psi^{(1)}$ with $\Psi^{(2)} = \Psi_0^{(2)} + \Psi_1^{(2)} + \Psi_2^{(2)}$, and there are remaining terms R_1, R_2, $\widetilde{R}_1$, $\widetilde{R}_2$, and $\widetilde{R}_3$. We now calculate R_1 and R_2. These terms involve a triple operator product of V's and the results depend on V. We choose

$$V(z) = i\sqrt{\frac{2}{\alpha'}}\,\partial X(z)\,, \tag{4.45}$$

which is *exactly* marginal. With this choice of V, the triple operator product of V's on $\mathcal{W}_{n-1}$ is

$$\begin{aligned} V(z_1)V(z_2)V(z_3) &= G_{n-1}(z_1-z_2)\,V(z_3) + G_{n-1}(z_1-z_3)\,V(z_2) + G_{n-1}(z_2-z_3)\,V(z_1) \\ &\quad + :V(z_1)V(z_2)V(z_3):\,, \end{aligned} \tag{4.46}$$

where G_{n-1} is the propagator on $\mathcal{W}_{n-1}$:

$$G_{n-1}(z) = \frac{\pi^2}{n^2}\left[\sin\frac{\pi z}{n}\right]^{-2} = \frac{1}{z^2} + O(z^0)\,. \tag{4.47}$$

The normal-ordered term in (4.46) does not contribute in the calculations of R_1 and R_2 in the limit $\epsilon \to 0$. The term with $V(1)$ and $V(1+2\epsilon)$ contracted in R_1 cancels $\widetilde{R}_1$:

$$\lim_{\epsilon\to 0}\left[\int_{2\epsilon}^{1} dt_2\, G_{1+t_2+2\epsilon}(2\epsilon)\,\langle\, f\circ\phi(0)\,c(1)\,c(1+2\epsilon)\,\mathcal{B}\,cV(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}} + \widetilde{R}_1\right] = 0\,. \tag{4.48}$$

The remaining two terms are finite in the limit $\epsilon \to 0$:

$$\begin{aligned} &\lim_{\epsilon\to 0}\Bigg[\int_{2\epsilon}^{1} dt_2\, G_{1+t_2+2\epsilon}(t_2)\,\langle\, f\circ\phi(0)\,cV(1)\,c(1+2\epsilon)\,\mathcal{B}\,c(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}} \\ &\qquad + \int_{2\epsilon}^{1} dt_2\, G_{1+t_2+2\epsilon}(t_2+2\epsilon)\,\langle\, f\circ\phi(0)\,c(1)\,cV(1+2\epsilon)\,\mathcal{B}\,c(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}}\Bigg] \\ &= \frac{3}{2}\,\langle\, f\circ\phi(0)\,c\partial cV(1)\,\rangle_{\mathcal{W}_1}. \end{aligned} \tag{4.49}$$

We therefore have

$$\lim_{\epsilon\to 0}\left[R_1 + \widetilde{R}_1\right] = \frac{3}{2}\,\langle\, f\circ\phi(0)\,c\partial cV(1)\,\rangle_{\mathcal{W}_1}. \tag{4.50}$$

The calculation of R_2 is parallel, and we obtain

$$\lim_{\epsilon\to 0}\left[R_2 + \widetilde{R}_2\right] = \frac{3}{2}\,\langle\, f\circ\phi(0)\,c\partial cV(1)\,\rangle_{\mathcal{W}_1}. \tag{4.51}$$

The sum of the five remaining terms vanishes in the limit $\epsilon \to 0$:

$$\lim_{\epsilon\to 0} \Big[R_1 + R_2 + \widetilde{R}_1 + \widetilde{R}_2 + \widetilde{R}_3 \Big] = 0\,. \tag{4.52}$$

We have thus shown

$$\lim_{\epsilon\to 0} \langle\, \phi, Q_B\, [\, \Psi_0^{(3)} + \Psi_1^{(3)}\,] + \Psi^{(1)} * [\, \Psi_0^{(2)} + \Psi_1^{(2)}\,] + [\, \Psi_0^{(2)} + \Psi_1^{(2)}\,] * \Psi^{(1)}\, \rangle = 0 \tag{4.53}$$

and

$$\lim_{\epsilon\to 0} \langle\, \phi, Q_B\, [\, \Psi_0^{(3)} + \Psi_1^{(3)} + \Psi_2^{(3)}\,] + \Psi^{(1)} * \Psi^{(2)} + \Psi^{(2)} * \Psi^{(1)}\, \rangle = 0\,. \tag{4.54}$$

The sum of the five terms did not have to vanish in the limit $\epsilon \to 0$, but it had to be BRST exact to satisfy the equation of motion by adding a counterterm. In particular, the coefficient in front of $\langle\, f \circ \phi(0)\, c\partial c V(1)\, \rangle_{\mathcal{W}_1}$ had to vanish. We found that $\widetilde{R}_3$ from $\Psi_1^{(3)}$ is nontrivially canceled by contributions from $\Psi_0^{(3)}$.

Let us next study the divergent terms of $\Psi_0^{(3)}$. The triple operator product of V's in (4.33) can be written as follows:

$$\begin{aligned} & V(1)\, V(1+t_1)\, V(1+t_1+t_2) \\ & = G_{1+t_1+t_2}(t_2)\, V(1) + G_{1+t_1+t_2}(t_1)\, V(1+t_1+t_2) \\ & \quad + G_{1+t_1+t_2}(t_1+t_2)\, V(1+t_1) + :V(1)\, V(1+t_1)\, V(1+t_1+t_2):\, . \end{aligned} \tag{4.55}$$

Note that no further divergence appears when remaining operators collide. The contribution from the normal-ordered product in the last line is obviously finite. The divergent terms from the first two terms on the right-hand side are canceled by the divergent terms from $\Psi_1^{(3)}$ and $\Psi_2^{(3)}$. The contribution from the third term on the right-hand side is

$$\begin{aligned} & \int_{2\epsilon}^{1} dt_1 \int_{2\epsilon}^{1} dt_2\, \left(\frac{\pi}{t_1+t_2+2} \right)^2 \left[\sin \frac{\pi\,(t_1+t_2)}{t_1+t_2+2} \right]^{-2} \\ & \qquad \times \langle\, f \circ \phi(0)\, c(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, c(1+t_1+t_2)\, \rangle_{\mathcal{W}_{1+t_1+t_2}}. \end{aligned} \tag{4.56}$$

This contains a divergent term $-\ln(4\epsilon)\, \langle\, f \circ \phi(0)\, cV(1)\, \rangle_{\mathcal{W}_1}$, which comes from the most singular term $1/(t_1{+}t_2)^2$ in the region where t_1 and t_2 are simultaneously of $O(\epsilon)$. Note that the divergent term is proportional to $\Psi^{(1)}$ and thus BRST closed, as expected. Therefore, if we define

$$\Psi^{(3)} = \lim_{\epsilon\to 0} \Big[\, \Psi_0^{(3)} + \Psi_1^{(3)} + \Psi_2^{(3)} + \Psi_3^{(3)} \Big]\,, \tag{4.57}$$

where

$$\Psi_3^{(3)} = \ln(4\epsilon)\, \Psi^{(1)}\,, \tag{4.58}$$

$\Psi^{(3)}$ is finite and satisfies the equation of motion:

$$\langle\, \phi,\, Q_B \Psi^{(3)} + \Psi^{(1)} * \Psi^{(2)} + \Psi^{(2)} * \Psi^{(1)}\, \rangle = 0\,. \tag{4.59}$$

An explicit form of $\Psi^{(3)}$ is given by

$$\begin{aligned}
\langle\, \phi, \Psi^{(3)} \,\rangle = \lim_{\epsilon\to 0} \Big[& \int_{2\epsilon}^{1} dt_1 \int_{2\epsilon}^{1} dt_2 \, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, cV(1+t_1+t_2)\, \rangle_{\mathcal{W}_{1+t_1+t_2}} \\
& - \frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_1 \, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, c(1+t_1+\epsilon)\, \rangle_{\mathcal{W}_{1+t_1+2\epsilon}} \\
& - \frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_2 \, \langle\, f\circ\phi(0)\, c(1+\epsilon)\, \mathcal{B}\, cV(1+t_2+2\epsilon)\, \rangle_{\mathcal{W}_{1+t_2+2\epsilon}} \\
& + \ln(2\epsilon) \int_{2\epsilon}^{1} dt_1 \, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\; Q_B\cdot[\,\mathcal{B}\, c(1+t_1)\,]\, \rangle_{\mathcal{W}_{1+t_1}} \\
& + \ln(2\epsilon) \int_{2\epsilon}^{1} dt_2 \, \langle\, f\circ\phi(0)\; Q_B\cdot[\,\mathcal{B}\, c(1)\,]\; \mathcal{B}\, cV(1+t_2)\, \rangle_{\mathcal{W}_{1+t_2}} \\
& + \ln(4\epsilon)\, \langle\, f\circ\phi(0)\, cV(1)\, \rangle_{\mathcal{W}_1} \Big] \,.
\end{aligned} \tag{4.60}$$

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